Y2P1 XMQs and MS

(Total: 30 marks)

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1. P2_Specimen  Q14.  8 marks - Y2P1 Algebraic methods

2. P2_2018      Q11.  7 marks - Y2P1 Algebraic methods

3. P1_2020      Q16.  4 marks - Y2P1 Algebraic methods

4. P1_2021      Q15.  6 marks - Y2P1 Algebraic methods

5. P1_2022      Q7 .  5 marks - Y2P1 Algebraic methods
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14. (i) Kayden claims that

$$3^x \geqslant 2^x$$

Determine whether Kayden's claim is always true, sometimes true or never true, justifying your answer.

(2)

(ii) Prove that $\sqrt{3}$ is an irrational number.

(6)

Question	Scheme	Marks	AOs
14 (i)	For an explanation or statement to show when the claim $3^x \ldots 2^x$ fails This could be e.g. • when $x = -1$, $\frac{1}{3} < \frac{1}{2}$ or $\frac{1}{3}$ is not greater than or equal to $\frac{1}{2}$ • when $x < 0$, $3^x < 2^x$ or 3^x is not greater than or equal to 2^x	M1	2.3
	followed by an explanation or statement to show when the claim $3^x \ldots 2^x$ is true. This could be e.g. • $x = 2$, $9 \ldots 4$ or 9 is greater than or equal to 4 • when $x \ldots 0$, $3^x \ldots 2^x$ and a correct conclusion. E.g. • so the claim $3^x \ldots 2^x$ is sometimes true	A1	2.4
		(2)	
(ii)	Assume that $\sqrt{3}$ is a rational number So $\sqrt{3} = \frac{p}{q}$, where p and q integers, $q \neq 0$, and the HCF of p and q is 1	M1	2.1
	$\Rightarrow p = \sqrt{3}q \Rightarrow p^2 = 3q^2$	M1	1.1b
	$\Rightarrow$ p^2 is divisible by 3 and so p is divisible by 3	A1	2.2a
	So $p = 3c$, where c is an integer From earlier, $p^2 = 3q^2 \Rightarrow (3c)^2 = 3q^2$	M1	2.1
	$\Rightarrow q^2 = 3c^2 \Rightarrow$ q^2 is divisible by 3 and so q is divisible by 3	A1	1.1b
	As both p and q are both divisible by 3 then the HCF of p and q is not 1 This contradiction implies that $\sqrt{3}$ is an irrational number	A1	2.4
		(6)	
			(8 marks)

Question 14 Notes:	
(i)	
M1:	See scheme
A1:	See scheme
(ii)	
M1:	Uses a method of proof by contradiction by initially assuming that $\sqrt{3}$ is rational and expresses $\sqrt{3}$ in the form $\frac{p}{q}$, where p and q are correctly defined.
M1:	Writes $\sqrt{3} = \frac{p}{q}$ and rearranges to make p^2 the subject
A1:	Uses a logical argument to prove that p is divisible by 3
M1:	Uses the result that p is divisible by 3, (to construct the initial stage of proving that q is also divisible by 3), by substituting $p = 3c$ into their expression for p^2
A1:	Hence uses a correct argument, in the same way as before, to deduce that q is also divisible by 3
A1:	Completes the argument (as detailed on the scheme) that $\sqrt{3}$ is irrational. **Note:** All the previous 5 marks need to be scored in order to obtain the final A mark.

11.

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$$

(a) Find the values of the constants A, B and C.

(4)

$$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)} \qquad x > 3$$

(b) Prove that $f(x)$ is a decreasing function.

(3)

Question	Scheme	Marks	AOs
11	$\dfrac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A+\dfrac{B}{(x-3)}+\dfrac{C}{(1-2x)}$		
(a) Way 1	$1+11x-6x^2 \equiv A(1-2x)(x-3)+B(1-2x)+C(x-3) \Rightarrow B=..., C=...$	M1	2.1
	$A=3$	B1	1.1b
	Uses substitution or compares terms to find either $B=...$ or $C=...$	M1	1.1b
	$B=4$ and $C=-2$ which have been found using a correct identity	A1	1.1b
		(4)	
(a) Way 2	{long division gives} $\dfrac{1+11x-6x^2}{(x-3)(1-2x)} \equiv 3+\dfrac{-10x+10}{(x-3)(1-2x)}$		
	$-10x+10 \equiv B(1-2x)+C(x-3) \Rightarrow B=..., C=...$	M1	2.1
	$A=3$	B1	1.1b
	Uses substitution or compares terms to find either $B=...$ or $C=...$	M1	1.1b
	$B=4$ and $C=-2$ which have been found using $-10x+10 \equiv B(1-2x)+C(x-3)$	A1	1.1b
		(4)	
(b)	$\text{f}(x)=3+\dfrac{4}{(x-3)}-\dfrac{2}{(1-2x)}$ $\{=3+4(x-3)^{-1}-2(1-2x)^{-1}\}$; $x>3$		
	$\text{f}'(x)=-4(x-3)^{-2}-4(1-2x)^{-2}$ $\left\{=-\dfrac{4}{(x-3)^2}-\dfrac{4}{(1-2x)^2}\right\}$	M1 A1ft	2.1 1.1b
	Correct $\text{f}'(x)$ **and** as $(x-3)^2>0$ and $(1-2x)^2>0$, then $\text{f}'(x)=-(+\text{ve})-(+\text{ve})<0$, so $\text{f}(x)$ is a decreasing function	A1	2.4
		(3)	
			(7 marks)

Notes for Question 11

(a)	
M1:	**Way 1:** Uses a correct identity $1+11x-6x^2 \equiv A(1-2x)(x-3)+B(1-2x)+C(x-3)$ in a complete method to find values for *B* and *C*. **Note:** Allow one slip in copying $1+11x-6x^2$ **Way 2:** Uses a correct identity $-10x+10 \equiv B(1-2x)+C(x-3)$ (which has been found from long division) in a complete method to find values for *B* and *C*
B1:	$A=3$
M1:	Attempts to find the value of either *B* or *C* from their identity This can be achieved by ***either*** substituting values into their identity ***or*** by comparing coefficients and solving the resulting equations simultaneously
A1:	See scheme
Note:	**Way 1:** Comparing terms: $x^2: \; -6=-2A; \quad x: \; 11=7A-2B+C; \quad \text{constant}: \; 1=-3A+B-3C$ **Way 1:** Substituting: $x=3: \; -20=-5B \Rightarrow B=4; \; x=\frac{1}{2}: \; 5=-\frac{5}{2}C \Rightarrow C=-2$
Note:	**Way 2:** Comparing terms: $x: \; -10=-2B+C; \quad \text{constant}: \; 10=B-3C$ **Way 2:** Substituting: $x=3: \; -20=-5B \Rightarrow B=4; \; x=\frac{1}{2}: \; 5=-\frac{5}{2}C \Rightarrow C=-2$

Note:	$A=3,\ B=4,\ C=-2$ from no working scores M1B1M1A1
Note:	The final A1 mark is effectively dependent upon both M marks

Notes for Question 11 Continued	
(a) ctd	
Note:	Writing $1+11x-6x^2 \equiv B(1-2x)+C(x-3) \Rightarrow B=4,\ C=-2$ will get 1st M0, 2nd M1, 1st A0
Note:	**Way 1:** You can imply a correct identity $1+11x-6x^2 \equiv A(1-2x)(x-3)+B(1-2x)+C(x-3)$ from seeing $\dfrac{1+11x-6x^2}{(x-3)(1-2x)} \equiv \dfrac{A(1-2x)(x-3)+B(1-2x)+C(x-3)}{(x-3)(1-2x)}$
Note:	**Way 2:** You can imply a correct identity $-10x+10 \equiv B(1-2x)+C(x-3)$ from seeing $\dfrac{-10x+10}{(x-3)(1-2x)} \equiv \dfrac{B(1-2x)+C(x-3)}{(x-3)(1-2x)}$
(b)	
M1:	Differentiates to give $\{\mathrm{f}'(x)=\}\ \pm\lambda(x-3)^{-2} \pm \mu(1-2x)^{-2};\ \lambda,\ \mu \neq 0$
A1ft:	$\mathrm{f}'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2}$, which can be simplified or un-simplified
Note:	Allow A1ft for $\mathrm{f}'(x) = -(\text{their } B)(x-3)^{-2} + (2)(\text{their } C)(1-2x)^{-2}$; $(\text{their } B), (\text{their } C) \neq 0$
A1:	$\mathrm{f}'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2}$ or $\mathrm{f}'(x) = -\dfrac{4}{(x-3)^2} - \dfrac{4}{(1-2x)^2}$ **and** a correct explanation e.g. $\mathrm{f}'(x) = -(+\text{ ve}) - (+\text{ ve}) < 0$, so $\mathrm{f}(x)$ is a decreasing {function}
Note:	The final A mark can be scored in part (b) from an incorrect $A=...$ or from $A=0$ or no value of A found in part (a)

Notes for Question 11 Continued - Alternatives

(a)

Note: Be aware of the following alternative solutions, by initially dividing by $"(x-3)"$ or $"(1-2x)"$

- $$\frac{1+11x-6x^2}{"(x-3)"(1-2x)} \equiv \frac{-6x-7}{(1-2x)} - \frac{20}{(x-3)(1-2x)} \equiv 3 - \frac{10}{(1-2x)} - \frac{20}{(x-3)(1-2x)}$$

$$\frac{20}{(x-3)(1-2x)} \equiv \frac{D}{(x-3)} + \frac{E}{(1-2x)} \Rightarrow 20 \equiv D(1-2x) + E(x-3) \Rightarrow D = -4,\ E = -8$$

$$\Rightarrow 3 - \frac{10}{(1-2x)} - \left(\frac{-4}{(x-3)} + \frac{-8}{(1-2x)}\right) \equiv 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)};\ A=3,\ B=4,\ C=-2$$

- $$\frac{1+11x-6x^2}{(x-3)"(1-2x)"} \equiv \frac{3x-4}{(x-3)} + \frac{5}{(x-3)(1-2x)} \equiv 3 + \frac{5}{(x-3)} + \frac{5}{(x-3)(1-2x)}$$

$$\frac{5}{(x-3)(1-2x)} \equiv \frac{D}{(x-3)} + \frac{E}{(1-2x)} \Rightarrow 5 \equiv D(1-2x) + E(x-3) \Rightarrow D = -1,\ E = -2$$

$$\Rightarrow 3 + \frac{5}{(x-3)} + \left(\frac{-1}{(x-3)} + \frac{-2}{(1-2x)}\right) \equiv 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)};\ A=3,\ B=4,\ C=-2$$

(b)

Alternative Method 1:

$$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)},\ x>3 \Rightarrow f(x) = \frac{1+11x-6x^2}{-2x^2+7x-3};\ \begin{Bmatrix} u = 1+11x-6x^2 & v = -2x^2+7x-3 \\ u' = 11-12x & v' = -4x+7 \end{Bmatrix}$$

$f'(x) = \dfrac{(-2x^2+7x-3)(11-12x)-(1+11x-6x^2)(-4x+7)}{(-2x^2+7x-3)^2}$	Uses quotient rule to find $f'(x)$	M1
	Correct differentiation	A1
$f'(x) = \dfrac{-20((x-1)^2+1)}{(-2x^2+7x-3)^2}$ and a correct explanation, e.g. $f'(x) = -\dfrac{(+\text{ ve})}{(+\text{ ve})} < 0$, so $f(x)$ is a decreasing {function}		A1

Alternative Method 2:

Allow M1A1A1 for the following solution:

Given $f(x) = 3 + \dfrac{4}{(x-3)} - \dfrac{2}{(1-2x)} = 3 + \dfrac{4}{(x-3)} + \dfrac{2}{(2x-1)}$

as $\dfrac{4}{(x-3)}$ decreases when $x>3$ and $\dfrac{2}{(2x-1)}$ decreases when $x>3$

then $f(x)$ is a decreasing {function}

16. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

Question	Scheme	Marks	AOs
16	Sets up the contradiction and factorises: There are positive integers *p* and *q* such that $(2p+q)(2p-q)=25$	M1	2.1
	If true then $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ or $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ **Award for deducing either of the above statements**	M1	2.2a
	Solutions are $p=6.5, q=12$ or $p=2.5, q=0$ Award for one of these	A1	1.1b
	This is a contradiction as there are no integer solutions hence there are no positive integers *p* and *q* such that $4p^2-q^2=25$	A1	2.1
		(4)	
	(4 marks)		
Notes:			

M1: For the key step in setting up the contradiction and factorising

M1: For deducing that for *p* and *q* to be integers then either $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ or $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ must be true.

Award for deducing either of the above statements.

You can ignore any reference to $\begin{matrix}2p+q=1\\2p-q=25\end{matrix}$ as this could not occur for positive *p* and *q*.

A1: For correctly solving one of the given statements,

For $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ candidates only really need to proceed as far as $p=6.5$ to show the contradiction.

For $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ candidates only really need to find either *p* or *q* to show the contradiction.

Alt for $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ candidates could state that $2p+q\neq 2p-q$ if p,q are positive integers.

A1: For a complete and rigorous argument with both possibilities and a correct conclusion.

Question	Scheme	Marks	AOs
16 Alt 1	Sets up the contradiction, attempts to make q^2 or $4p^2$ the subject and states that either $4p^2$ is even(*) , or that q^2 (or *q*) is odd (**) Either There are positive integers *p* and *q* such that $4p^2 - q^2 = 25 \Rightarrow q^2 = 4p^2 - 25$ with * or ** Or There are positive integers *p* and *q* such that $4p^2 - q^2 = 25 \Rightarrow 4p^2 = q^2 + 25$ with * or **	M1	2.1
	Sets $q = 2n \pm 1$ and expands $(2n \pm 1)^2 = 4p^2 - 25$	M1	2.2a
	Proceeds to an expression such as $4p^2 = 4n^2 + 4n + 26 = 4(n^2 + n + 6) + 2$ $4p^2 = 4n^2 + 4n + 26 = 4(n^2 + n) + \frac{13}{2}$ $p^2 = n^2 + n + \frac{13}{2}$	A1	1.1b
	States This is a contradiction as $4p^2$ must be a multiple of 4 Or p^2 must be an integer And concludes there are no positive integers *p* and *q* such that $4p^2 - q^2 = 25$	A1	2.1
		(4)	

Alt 2

An approach using odd and even numbers is unlikely to score marks.

To make this consistent with the Alt method, score

M1: Set up the contradiction and start to consider one of the cases below where *q* is odd, $m \neq n$.

Solutions using the same variable will score no marks.

M1: Set up the contradiction and start to consider BOTH cases below where *q* is odd, $m \neq n$.

No requirement for evens

A1: Correct work and deduction for one of the two scenarios where *q* is odd

A1: Correct work and deductions for both scenarios where *q* is odd with a final conclusion

Options	Example of Calculation	Deduction
p (even) *q* (odd)	$4p^2 - q^2 = 4\times(2m)^2 - (2n+1)^2 = 16m^2 - 4n^2 - 4n - 1$	One less than a multiple of 4 so cannot equal 25
p (odd) *q* (odd)	$4p^2 - q^2 = 4\times(2m+1)^2 - (2n+1)^2 = 16m^2 + 16m - 4n^2 - 4n + 3$	Three more than a multiple of 4 so cannot equal 25

15. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n + 1)^3 > 3^n$$

(2)

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question	Scheme	Marks	AOs
15(i)	$n=1,\ 2^3=8,\ 3^1=3,\ (8>3)$ $n=2,\ 3^3=27,\ 3^2=9,\ (27>9)$ $n=3,\ 4^3=64,\ 3^3=27,\ (64>27)$ $n=4,\ 5^3=125,\ 3^4=81,\ (125>81)$	M1	2.1
	So if $n \leqslant 4, n \in \mathbb{N}$ then $(n+1)^3 > 3^n$	A1	2.4
		(2)	
(ii)	Begins the proof by negating the statement. "Let m be odd " or "Assume m is not even"	M1	2.4
	Set $m=(2p\pm1)$ and attempt $m^3+5=(2p\pm1)^3+5=...$	M1	2.1
	$=8p^3+12p^2+6p+6$ AND deduces even	A1	2.2a
	Completes proof which requires reason and conclusion • reason for $8p^3+12p^2+6p+6$ being even • acceptable statement such as "this is a contradiction so if m^3+5 is odd then m must be even"	A1	2.4
		(4)	
			(6 marks)
	Notes		

(i)

M1: A full and rigorous argument that uses all of $n = 1, 2, 3$ and 4 in an attempt to prove the given result. Award for attempts at both $(n+1)^3$ and 3^n for **ALL** values with at least 5 of the 8 values correct.

There is no requirement to compare their sizes, for example state that $27>9$

Extra values, say $n = 0$, may be ignored

A1: Completes the proof with no errors and an appropriate/allowable conclusion.

This requires

- all the values for $n=1, 2, 3$ and 4 correct. Ignore other values
- all pairs compared correctly
- a minimal conclusion. Accept ✓ or hence proven for example

(ii)

M1: Begins the proof by negating the statement. See scheme

This cannot be scored if the candidate attempts m both odd and even

M1: For the key step in setting $m=2p\pm1$ and attempting to expand $(2p\pm1)^3+5$

Award for a 4 term cubic expression.

A1: Correctly reaches $(2p+1)^3+5=8p^3+12p^2+6p+6$ and **states** even.

Alternatively reaches $(2p-1)^3+5=8p^3-12p^2+6p+4$ and **states** even.

A1: A full and complete argument that completes the contradiction proof. See scheme.

(1) **A reason** why the expression $8p^3+12p^2+6p+6$ or $8p^3-12p^2+6p+4$ is even

Acceptable reasons are

- all terms are even
- sight of a factorised expression E.g. $8p^3-12p^2+6p+4=2(4p^3-6p^2+3p+2)$

(2) Acceptable concluding statement

Acceptable concluding statements are

- "this is a contradiction, so if m^3+5 is odd then m is even"
- "this is contradiction, so proven."
- "So if m^3+5 is odd them m is even"

S.C If the candidate misinterprets the demand and does not use proof by contradiction but states a counter example to the statement “if m^3+5 is odd then m must be even” such as when $m=\sqrt[3]{2}$ then they can score special case mark B1

7. (i) Given that p and q are integers such that

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

(3)

(ii) Given that x and y are integers such that

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

(2)

Question	Scheme	Marks	AOs
7 (i)	For setting up the contradiction: There exists integers p and q such that pq is even and both p and q are odd	B1	2.5
	For example, sets $p = 2m+1$ and $q = 2n+1$ and then attempts $pq = (2m+1)(2n+1) = ...$	M1	1.1b
	Obtains $pq = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$ $= 2(2mn + m + n) + 1$ States that this is odd, giving a contradiction so " if pq is even, then at least one of p and q is even" *	A1*	2.1
		(3)	
(ii)			
	$(x+y)^2 < 9x^2 + y^2 \Rightarrow 2xy < 8x^2$	M1	2.2a
	States that as $x < 0 \Rightarrow 2y > 8x$ $\Rightarrow y > 4x$ *	A1*	2.1
		(2)	
			(5 marks)

Notes:

(i)

B1: For using the "correct"/ allowable language in setting up the contradiction.
Expect to see a minimum of

- "assume" or "let" or "there is " or other similar words
- "*pq* is even" and "*p* and *q* are (both) odd"

M1: Uses a correct algebraic form for *p* and *q* and attempting to multiply.
Allow any correct form so $p = 2n+1$ and $q = 2m+3$ would be fine to use
Different variables must be used for *p* and *q*, so $p = 2n+1$ and $q = 2n-1$ would be M0

A1*: Full argument .
This requires (1) a correct calculation for their *pq*
(2) a correct reason and conclusion that it is odd
E.g. $(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1 = \text{odd}$
E.g. $(2m-1)(2n+1) = 4mn + 2m - 2n - 1 = \text{even} + \text{even} - \text{even} - 1 = \text{odd}$
and (3) a minimal statement implying that they have proven what was required which could be QED, proven etc

Note that B0 M1 A1 is possible

(ii)

M1: For multiplying out and cancelling terms before proceeding to a correct intermediate line such as $2xy < 8x^2$ o.e. such as $2x(4x-y) > 0$

A1*: Full and rigorous proof with reason shown as to why inequality reverses. The point at which it reverses must be correct and a correct reason given

See scheme

Alt: $2xy < 8x^2 \Rightarrow xy - 4x^2 < 0 \Rightarrow x(y-4x) < 0$

as $x < 0$, $(y-4x) > 0 \Rightarrow y > 4x$ scores M1 A1

So, the following should be scored M1 A0 as line 3 is incorrect

$$2xy - 8x^2 < 0$$
$$\Rightarrow 2xy < 8x^2$$
$$\Rightarrow y < 4x$$
$$\Rightarrow y > 4x \text{ as } x < 0$$

There should be no incorrect lines in their proof