The Silver Rectangle

Why are the dimensions of A4 paper so weird?

The A4 paper system (officially known as ISO 216) has some strange-looking measurements, but there are very good mathematical reasons for them.

By measuring this sheet of paper, fill in the blanks in the following statements:
Note: for the purposes of these calculations, take 'length' to be the size of the longest edge.

A4 paper is __________ *mm* long and __________*mm* wide.

If you fold A4 paper in half, short side to short side, the resulting size is A5.

A5 paper is __________ *mm* long and __________ *mm* wide.

If you put two pieces of A4 paper together, long side to long side, you get A3.

A3 paper is __________ *mm* long and __________ *mm* wide.

Identify the pattern and calculate the length and width of A2, A1 and A0 paper:

Paper size	Length (mm)	Width (mm)	Aspect Ratio	Area (mm^2)
A0				
A1				
A2				
A3				
A4				
A5				

"Aspect Ratio" is a measure of the shape of the paper: $Aspect\ Ratio = \frac{Length}{Width}$

"Area" is the size of the paper: $Area = Length \times Width$

Complete the table above, finding the aspect ratio and the area of each size.

What do you notice about the aspect ratio?
Hint: What is the area scale factor between one paper size and the next?

What do you notice about the area of the largest paper size?
Hint: recall that $1m^2 = 1000mm \times 1000mm = 1{,}000{,}000mm^2$

The Silver Rectangle SOLUTIONS

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By measuring this sheet of paper, fill in the blanks in the following statements:
Note: for the purposes of these calculations, take 'length' to be the size of the longest edge.

A4 paper is 297 mm long and 210 mm wide.

If you fold A4 paper in half, short side to short side, the resulting size is A5.

A5 paper is 210 mm long and 148 mm wide.

If you put two pieces of A4 paper together, long side to long side, you get A3.

A3 paper is 420 mm long and 297 mm wide.

Identify the pattern and calculate the length and width of A2, A1 and A0 paper:

Paper size	Length (mm)	Width (mm)	Aspect Ratio	Area (mm^2)
A0	**1189**	**841**	1.413793103	999949
A1	**841**	**594**	1.415824916	499554
A2	**594**	**420**	1.414285714	249480
A3	420	297	1.414141414	124740
A4	297	210	1.414285714	62370
A5	210	148	1.418918919	31080

"Aspect Ratio" is a measure of the shape of the paper: $Aspect\ Ratio = \frac{Length}{Width}$

"Area" is the size of the paper: $Area = Length \times Width$

Complete the table above, finding the aspect ratio and the area of each size.

What do you notice about the aspect ratio?
Hint: What is the area scale factor between one paper size and the next?
$\sqrt{2} \approx 1.414213562$. All the numbers are almost the same, and close to $\sqrt{2}$.
What do you notice about the area of the largest paper size?
Hint: recall that $1m^2 = 1000mm \times 1000mm = 1{,}000{,}000mm^2$
The area of A0 paper is very nearly $1m^2$. It is $51mm^2$ smaller, or this much: 

The Silver Rectangle: Origins

Generating the A4 paper system

The A4 paper system (officially known as ISO 216) has some strange-looking measurements, but there are very good mathematical reasons for them. When deciding what dimensions to use, the following requirements were decided upon:

- Aspect ratio must be maintained across all paper sizes (that is, halving one sheet produces a sheet with the same shape as the previous one)
- The area and the weight of individual sheets must be 'nice' values.

The first requirement is the most clearly defined, so we'll start with that.

Let's call the length and width of our first piece of paper L_1 and W_1. We want to be able to halve this (that is, fold in two short side to short side) and generate a sheet with dimensions in the same ratio. That is:

$$\frac{L_1}{W_1} = \frac{L_2}{W_2}$$

Now, since folding short side to short side means the width of the original becomes the length of the new sheet:

$$L_2 = W_1$$

And because the length halved becomes the new width:

$$W_2 = \frac{L_1}{2}$$

This means we can write the original equation as:

$$\frac{L_1}{W_1} = \frac{W_1}{\frac{L_1}{2}} = \frac{2W_1}{L_1}$$

Rearranging gives:

$$(L_1)^2 = 2(W_1)^2 \quad \Rightarrow \quad \frac{(L_1)^2}{(W_1)^2} = 2 \quad \Rightarrow \quad \frac{L_1}{W_1} = \sqrt{2}$$

So now we have our aspect ratio. The length divided by the width must equal $\sqrt{2}$.

The second condition is more vague, and by interpreting this differently, two or three paper systems can be (and have been) developed. Investigate the 'B' and 'C' series of paper if you're interested. However, considering the fact that paper density is measured in g/m^2 (grams per square metre), relating the size of paper directly to a square metre would be useful. So what would the dimensions have to be for the area to be exactly $1m^2$?

$$\frac{L}{W} = \sqrt{2} \quad and \quad LW = 1$$

Rearranging and substituting:

$$\frac{L}{\sqrt{2}} = W \quad \Rightarrow \quad \frac{L^2}{\sqrt{2}} = 1 \quad \Rightarrow \quad L^2 = \sqrt{2} \quad \Rightarrow \quad L = \sqrt[4]{2} \approx 1.189207115$$

$$W = \frac{1}{L} = \frac{1}{\sqrt[4]{2}} \approx 0.84089641525$$

Rounding to the nearest mm gives: $L = 1189mm \quad W = 841mm$. This is size $A0$.

By using $W_2 = \frac{L_1}{2}$ and $L_2 = W_1$, and rounding down to the nearest mm where necessary, we can generate size $A1$: $L = 841mm \quad W = 594mm$. Continuing this, can you find A4?