Practice examination questions for calculus

1. Jan 2005

4 For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(a) $\int_2^{\infty} 8x^{-3}\,dx$; *(3 marks)*

(b) $\int_2^{\infty} (8x^{-3}+1)\,dx$; *(1 mark)*

(c) $\int_2^{\infty} 8x^{-3}(x+1)\,dx$. *(3 marks)*

2. June 2005 (first principle)

4 The function f is defined for all real values of x by

$$f(x) = x^3 + x$$

(a) Express $f(2+h) - f(2)$ in the form

$$ph + qh^2 + rh^3$$

where p, q and r are integers. *(5 marks)*

(b) Use your answer to part (a) to find the value of $f'(2)$. *(2 marks)*

3. Jan 2006

2 (a) For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(i) $\int_0^9 \frac{1}{\sqrt{x}}\,dx$; *(3 marks)*

(ii) $\int_0^9 \frac{1}{x\sqrt{x}}\,dx$. *(3 marks)*

(b) Explain briefly why the integrals in part (a) are improper integrals. *(1 mark)*

4. June 2006 (first principle)

8 (a) The function f is defined for all real values of x by

$$f(x) = x^3 + x^2 - 1$$

(i) Express $f(1+h) - f(1)$ in the form

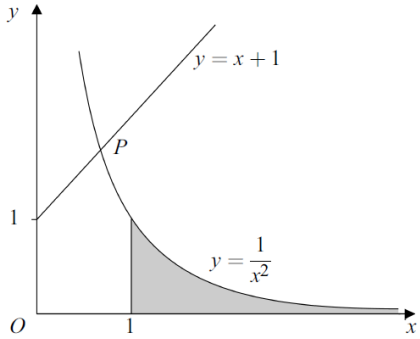
$$ph + qh^2 + rh^3$$

where p, q and r are integers. *(4 marks)*

(ii) Use your answer to part (a)(i) to find the value of $f'(1)$. *(2 marks)*

(b) The diagram shows the graphs of

$$y = \frac{1}{x^2} \quad \text{and} \quad y = x + 1 \quad \text{for} \quad x > 0$$



The graphs intersect at the point P.

(i) Show that the x-coordinate of P satisfies the equation $f(x) = 0$, where f is the function defined in part (a). *(1 mark)*

(ii) Taking $x_1 = 1$ as a first approximation to the root of the equation $f(x) = 0$, use the Newton–Raphson method to find a second approximation x_2 to the root. *(3 marks)*

(c) The region enclosed by the curve $y = \frac{1}{x^2}$, the line $x = 1$ and the x-axis is shaded on the diagram. By evaluating an improper integral, find the area of this region. *(3 marks)*

5. Jan 2007 (first principle)

7 The function f is defined for all real numbers by

$$f(x) = \sin\left(x + \frac{\pi}{6}\right)$$

(a) Find the general solution of the equation $f(x) = 0$. *(3 marks)*

(b) The quadratic function g is defined for all real numbers by

$$g(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2$$

It can be shown that $g(x)$ gives a good approximation to $f(x)$ for small values of x.

(i) Show that $g(0.05)$ and $f(0.05)$ are identical when rounded to four decimal places. *(2 marks)*

(ii) A chord joins the points on the curve $y = g(x)$ for which $x = 0$ and $x = h$. Find an expression in terms of h for the gradient of this chord. *(2 marks)*

(iii) Using your answer to part (b)(ii), find the value of $g'(0)$. *(1 mark)*

6. June 2007

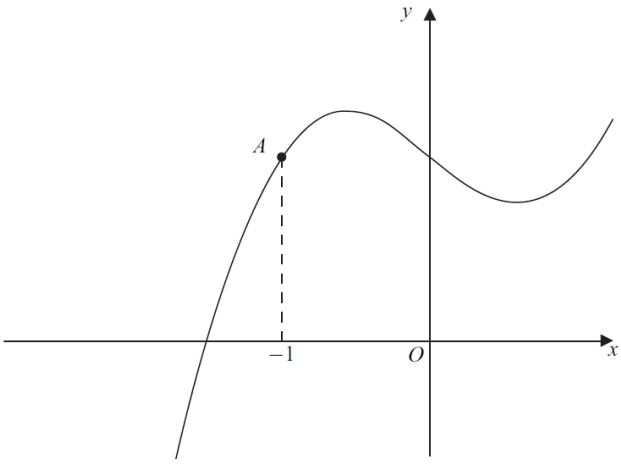
8 For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(a) $\int_0^1 (x^{\frac{1}{3}} + x^{-\frac{1}{3}})\,dx$; *(4 marks)*

(b) $\int_0^1 \frac{x^{\frac{1}{3}} + x^{-\frac{1}{3}}}{x}\,dx$. *(4 marks)*

7. Jan 2008(first principle)

Figure 1 (for use in Question 7)

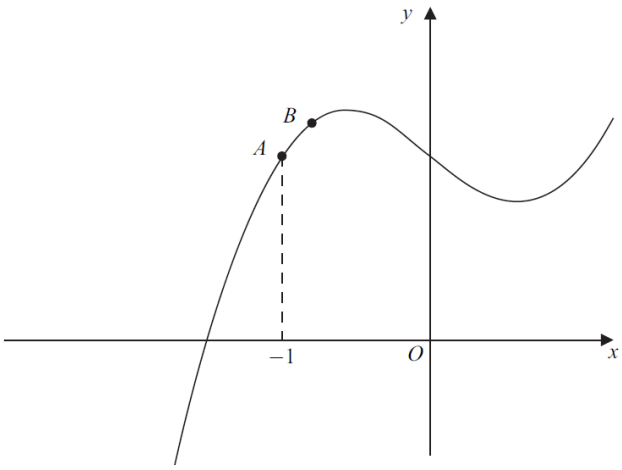


7 [**Figure 1**, printed on the insert, is provided for use in this question.]

The diagram shows the curve

$$y = x^3 - x + 1$$

The points A and B on the curve have x-coordinates -1 and $-1 + h$ respectively.



(a) (i) Show that the y-coordinate of the point B is

$$1 + 2h - 3h^2 + h^3$$ *(3 marks)*

(ii) Find the gradient of the chord AB in the form

$$p + qh + rh^2$$

where p, q and r are integers. *(3 marks)*

(iii) Explain how your answer to part (a)(ii) can be used to find the gradient of the tangent to the curve at A. State the value of this gradient. *(2 marks)*

(b) The equation $x^3 - x + 1 = 0$ has one real root, α.

(i) Taking $x_1 = -1$ as a first approximation to α, use the Newton-Raphson method to find a second approximation, x_2, to α. *(2 marks)*

(ii) On **Figure 1**, draw a straight line to illustrate the Newton-Raphson method as used in part (b)(i). Show the points $(x_2, 0)$ and $(\alpha, 0)$ on your diagram. *(2 marks)*

8. June 2008

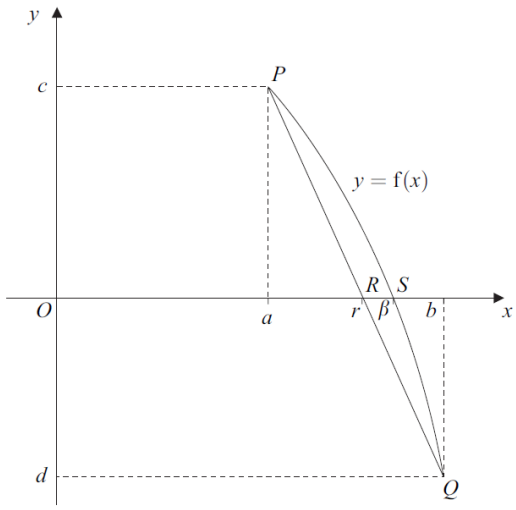
3 For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(a) $\displaystyle\int_9^{\infty} \frac{1}{\sqrt{x}}\,dx$; *(3 marks)*

(b) $\displaystyle\int_9^{\infty} \frac{1}{x\sqrt{x}}\,dx$. *(4 marks)*

9. Jan 2009

7 The points $P(a, c)$ and $Q(b, d)$ lie on the curve with equation $y = f(x)$. The straight line PQ intersects the x-axis at the point $R(r, 0)$. The curve $y = f(x)$ intersects the x-axis at the point $S(\beta, 0)$.



(a) Show that

$$r = a + c\left(\frac{b-a}{c-d}\right)$$ *(4 marks)*

(b) Given that

$$a = 2,\ b = 3 \text{ and } f(x) = 20x - x^4$$

(i) find the value of r; *(3 marks)*

(ii) show that $\beta - r \approx 0.18$. *(3 marks)*

10.Jan 2009

8 For each of the following improper integrals, find the value of the integral **or** explain why it does not have a value:

(a) $\int_1^{\infty} x^{-\frac{3}{4}}\,dx$; *(3 marks)*

(b) $\int_1^{\infty} x^{-\frac{5}{4}}\,dx$; *(3 marks)*

(c) $\int_{\cdot}^{\infty} (x^{-\frac{3}{4}} - x^{-\frac{5}{4}})\,dx$. *(1 mark)*

11.June 2009(first principle)

2 A curve has equation

$$y = x^2 - 6x + 5$$

The points A and B on the curve have x-coordinates 2 and $2 + h$ respectively.

(a) Find, in terms of h, the gradient of the line AB, giving your answer in its simplest form. *(5 marks)*

(b) Explain how the result of part **(a)** can be used to find the gradient of the curve at A. State the value of this gradient. *(3 marks)*

12.Jan 2010

5 (a) Explain why $\int_0^{\frac{1}{16}} x^{-\frac{1}{2}}\,dx$ is an improper integral. *(1 mark)*

(b) For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(i) $\int_0^{\frac{1}{16}} x^{-\frac{1}{2}}\,dx$; *(3 marks)*

(ii) $\int_0^{\frac{1}{16}} x^{-\frac{5}{4}}\,dx$. *(3 marks)*

13.June 2010(first principle)

5 A curve has equation $y = x^3 - 12x$.

The point A on the curve has coordinates $(2, -16)$.

The point B on the curve has x-coordinate $2 + h$.

(a) Show that the gradient of the line AB is $6h + h^2$. *(4 marks)*

(b) Explain how the result of part **(a)** can be used to show that A is a stationary point on the curve. *(2 marks)*

14.Jan 2011

2 (a) Find, in terms of p and q, the value of the integral $\int_p^q \frac{2}{x^3}\,dx$. *(3 marks)*

(b) Show that only one of the following improper integrals has a finite value, and find that value:

(i) $\int_0^2 \frac{2}{x^3}\,dx$;

(ii) $\int_2^\infty \frac{2}{x^3}\,dx$. *(3 marks)*

15. June 2011(first principle)

6 (a) Expand $(5+h)^3$. *(1 mark)*

(b) A curve has equation $y = x^3 - x^2$.

(i) Find the gradient of the line passing through the point (5, 100) and the point on the curve for which $x = 5 + h$. Give your answer in the form

$$p + qh + rh^2$$

where p, q and r are integers. *(4 marks)*

(ii) Show how the answer to part **(b)(i)** can be used to find the gradient of the curve at the point (5, 100). State the value of this gradient. *(2 marks)*

16. Jan 2012

2 Show that only one of the following improper integrals has a finite value, and find that value:

(a) $\int_8^\infty x^{-\frac{2}{3}}\,dx$;

(b) $\int_8^\infty x^{-\frac{4}{3}}\,dx$. *(5 marks)*

17. June 2012 (first principle)

2 A curve has equation $y = x^4 + x$.

(a) Find the gradient of the line passing through the point $(-2, 14)$ and the point on the curve for which $x = -2 + h$. Give your answer in the form

$$p + qh + rh^2 + h^3$$

where p, q and r are integers. *(5 marks)*

(b) Show how the answer to part **(a)** can be used to find the gradient of the curve at the point $(-2, 14)$. State the value of this gradient. *(2 marks)*

18. Jan 2013

4 Show that the improper integral $\int_{25}^{\infty} \frac{1}{x\sqrt{x}}\,dx$ has a finite value and find that value. *(4 marks)*

Answers

1. Jan 2005

4(a)	$\int x^{-3}\,dx = kx^{-2}\,(+c)$	M1		
	$x^{-n} \to 0$ as $x \to \infty$	M1		
	Improper integral has value 1	A1	3	
(b)	No value as x term tends to ∞	B1	1	OE
(c)	$\int x^{-2}\,dx = kx^{-1}\,(+c)$	M1		
	$x^{-1} \to 0$ as $x \to \infty$	m1		
	Improper integral has value 5	A1	3	
	Total		**7**	

2. June 2005

4(a)	$(2+h)^3 = 8 + ah + bh^2 + h^3$	M1		
	$(2+h)^3 = 8 + 12h + 6h^2 + h^3$	A1A1		A1 for each of a, b ; PI
	$f(2+h) - f(2) = 13h + 6h^2 + h^3$	m1A1F	5	Ft one coeff. wrong
(b)	Divide by h and let $h \to 0$	M1		NMS B1F
	$f'(2) = p = 13$	A1F	2	ft wrong value of p
	Total		**7**	

3. Jan 2006

3	One solution is $x = 10°$	B1		PI by general formula
	Use of sin 130° = sin 50°	M1		OE
	Second solution is $x = 30°$	A1		OE
	Introduction of $90n°$, or $360n°$ or $180n°$	M1		Or $\pi n/2$ or $2\pi n$ or πn
	GS $(10+90n)°, (30+90n)°$	A1√	5	OE; ft one numerical error or omission of 2nd soln
	Total		**5**	

4. June 2006

8(a)(i)	$(1+h)^3 = 1 + 3h + 3h^2 + h^3$	B1		
	$f(1+h) = 1 + 5h + 4h^2 + h^3$	M1A1√		PI; ft wrong coefficients
	$f(1+h) - f(1) = 5h + 4h^2 + h^3$	A1√	4	ft numerical errors
(ii)	Dividing by h	M1		
	$f'(1) = 5$	A1√	2	ft numerical errors
(b)(i)	$x^2(x+1) = 1$, hence result	B1	1	convincingly shown (AG)
(ii)		M1A1√		
	$x_2 = 1 - \frac{1}{5} = \frac{4}{5}$	A1√	3	ft c's value of f′(1)
(c)	Area $= \int_1^{\infty} x^{-2}\,dx$	M1		
	$\ldots = \left[-x^{-1}\right]_1^{\infty}$	M1		Ignore limits here
	$\ldots = 0 - -1 = 1$	A1	3	
	Total		**13**	

5. Jan 2007

7(a)	Particular solution, eg $-\frac{\pi}{6}$ or $\frac{5\pi}{6}$	B1		Degrees or decimals penalised in 3rd mark only
	Introduction of $n\pi$ or $2n\pi$	M1		
	GS $x = -\frac{\pi}{6} + n\pi$	A1F	3	OE(accept unsimplified); ft incorrect first solution
(b)(i)	f (0.05) ≈ 0.542 66 g (0.05) ≈ 0.542 68	B1 B1	 2	either value AWRT 0.5427 both values correct to 4DP
(ii)	$\frac{g(h) - g(0)}{h} = \frac{\sqrt{3}}{2} - \frac{1}{4}h$	M1A1	2	M1A0 if num error made
(iii)	As $h \to 0$ this gives $g'(0) = \frac{\sqrt{3}}{2}$	A1F	1	AWRT 0.866; ft num error
	Total		**8**	

6. June 2007

8(a)	$\int\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right) dx = \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} (+c)$	M1A1		M1 for adding 1 to index at least once
	$\int_0^1 \ldots = \left(\frac{3}{4} + \frac{3}{2}\right) - 0 = \frac{9}{4}$	m1A1	4	Condone no mention of limiting process; m1 if "– 0" stated or implied
(b)	Second term is $x^{-\frac{4}{3}}$	B1		
	Integral of this is $-3x^{-\frac{1}{3}}$	M1A1		M1 for correct index
	$x^{-\frac{1}{3}} \to \infty$ as $x \to 0$, so no value	E1	4	
	Total		**8**	

7. Jan 2008

7(a)(i)	$(-1 + h)^3 = -1 + 3h - 3h^2 + h^3$	B1		PI
	$y_B = (-1 + 3h - 3h^2 + h^3) + 1 - h + 1$	B1F		ft numerical error
	$\ldots = 1 + 2h - 3h^2 + h^3$	B1	3	convincingly shown (AG)
(ii)	Subtraction of 1 and division by h	M1M1		
	Gradient of chord $= 2 - 3h + h^2$	A1	3	
(iii)	As $h \to 0$, gr(chord) → gr(tgt) = 2	E1B1F	2	E0 if '$h = 0$' used; ft wrong value of p
(b)(i)	$x_2 = -1 - \frac{1}{2} = -1.5$	M1		
		A1F	2	ft wrong gradient
(ii)	Tangent at A drawn	M1		
	α and x_2 shown correctly	A1	2	dep't only on the last M1
	Total		**12**	

8. June 2008

3(a)	$\int x^{-\frac{1}{2}}\, dx = 2x^{\frac{1}{2}} (+c)$	M1A1		M1 for correct power in integral
	$x^{\frac{1}{2}} \to \infty$ as $x \to \infty$, so no value	E1	3	

(b)	$\int x^{-3/2}\,dx = -2x^{-1/2}(+c)$	M1A1		M1 for correct power in integral
	$x^{-1/2} \to 0$ as $x \to \infty$	E1		PI
	$\int_9^{\infty} x^{-3/2}\,dx = -2(0-\frac{1}{3}) = \frac{2}{3}$	A1	4	Allow A1 for correct answer even if not fully explained
	Total		**7**	

9. Jan 2009

7(a)	Use of similar triangles or algebra	M1		Some progress needed
	Correct relationship established	m1A1		eg $\frac{r-a}{c} = \frac{b-a}{c-d}$
	Hence result convincingly shown	A1	4	AG
(b)(i)	$c = f(a) = 24,\ d = f(b) = -21$	B1,B1		
	$r = \frac{38}{15}$ (≈ 2.5333)	B1F	3	Allow AWRT 2.53; ft small error
(ii)	$\beta = 20^{\frac{1}{3}} \approx 2.714(4)$	M1A1		Allow AWRT 2.71
	So $\beta - r \approx 0.181 \approx 0.18$ (AG)	A1	3	Allow only 2dp if earlier values to 3dp
	Total		**10**	

10.Jan 2009

8(a)	$\int x^{-\frac{3}{4}}\,dx = 4x^{\frac{1}{4}}\ (+c)$	M1A1		M1 if index correct
	This tends to ∞ as $x \to \infty$, so no value	A1F	3	ft wrong coefficient
(b)	$\int x^{-\frac{5}{4}}\,dx = -4x^{-\frac{1}{4}}\ (+c)$	M1A1		M1 if index correct
	$\int_1^{\infty} x^{-\frac{5}{4}}\,dx = 0 - (-4) = 4$	A1F	3	ft wrong coefficient
(c)	Subtracting 4 leaves ∞, so no value	B1F	1	ft if c has 'no value' in (a) but has a finite answer in (b)
	Total		**7**	

11.June 2009

2(a)	When $x = 2$, $y = -3$	B1		PI
	Use of $(2+h)^2 = 4 + 4h + h^2$	M1		
	Correct method for gradient	M1		
	Gradient $= \frac{-3-2h+h^2+3}{h} = -2+h$	A2,1	5	A1 if only one small error made
(b)	As h tends to 0,	E2,1		E1 for '$h = 0$'
	... the gradient tends to -2	B1F	3	dependent on at least E1 ft small error in (a)
	Total		**8**	

12.Jan 2010

5(a)	$x^{-1/2} \to \infty$ as $x \to 0$	E1	1	Condone "$x^{-1/2}$ has no value at $x = 0$"
(b)(i)	$\int x^{-1/2}\,dx = 2x^{1/2}\ (+c)$	M1A1		M1 for correct power of x
	$\int_0^{1/16} x^{-1/2}\,dx = \frac{1}{2}$	A1F	3	ft wrong coefficient of $x^{1/2}$

(ii)	$\int x^{-5/4}\,dx = -4x^{-1/4}\ (+c)$	M1A1		M1 for correct power of x
	$x^{-1/4} \to \infty$ as $x \to 0$, so no value	E1F	3	ft wrong coefficient of $x^{-1/4}$
	Total		**7**	

13.June 2010

5(a)	At B, $y = (2+h)^3 - 12(2+h)$	M1		with attempt to expand and simplify
	$= (8 + 12h + 6h^2 + h^3) - (24 + 12h)$ $(= -16 + 6h^2 + h^3)$	B1		correct expansion of $(2+h)^3$
	Grad $AB = \dfrac{(-16+6h^2+h^3)-(-16)}{(2+h)-2}$	m1		
	$= \dfrac{6h^2+h^3}{h} = 6h + h^2$	A1	4	convincingly shown (AG)
(b)	As $h \to 0$ this gradient $\to 0$ so gradient of curve at A is 0	E2,1	2	E1 for '$h = 0$'
	Total		**6**	

14.Jan 2011

2(a)	$\int 2x^{-3}\,dx = -x^{-2}\ (+c)$	M1A1		M1 for correct index
	$\int_p^q 2x^{-3}\,dx = p^{-2} - q^{-2}$	A1F	3	OE; ft wrong coefficient of x^{-2}
(b)(i)	As $p \to 0$, $p^{-2} \to \infty$, so no value	B1		
(ii)	As $q \to \infty$, $q^{-2} \to 0$, so value is ¼	M1A1F	3	ft wrong coefficient of x^{-2} or reversal of limits
	Total		**6**	

15.June 2011

6(a)	$(5+h)^3 = 125 + 75h + 15h^2 + h^3$	B1	1	Accept unsimplified coefficients
(b)(i)	$y(5+h) = 100 + 65h + 14h^2 + h^3$	B1F		PI; ft numerical error in (a)
	Use of correct formula for gradient	M1		
	Gradient is $65 + 14h + h^2$	A2,1F	4	A1 if one numerical error made; ft numerical error already penalised
(ii)	As $h \to 0$ this $\to 65$	E2,1F	2	E1 for '$h = 0$'; ft wrong values for p, q, r
	Total		**7**	

16.Jan 2012

2(a)	$\int x^{-2/3}\,dx = 3x^{1/3}\,(+c)$	B1		$kx^{\frac{1}{3}}$, $k \neq 0$ ie condone incorrect non-zero coefficient here
	$(3)x^{1/3} \to \infty$ as $x \to \infty$, so no finite value	E1		
(b)	$\int x^{-4/3}\,dx = -3x^{-1/3}\,(+c)$	M1		$\lambda x^{-1/3}$, $\lambda \neq 0$
		A1		$-3x^{-1/3}$ OE
	$\int_8^\infty x^{-4/3}\,dx = -3(0 - \frac{1}{2}) = \frac{3}{2}$	A1	5	CSO
	Total		**5**	

17.June 2012

2(a)	$y = x^4 + x$ $\{y(-2+h) =\}\quad (-2+h)^4 + (-2+h)$	M1		$(-2+h)^4 + (-2+h)$ PI
	$= h^4 - 8h^3 + 24h^2 - 32h + 16 - 2 + h$	B1		Correct expansion of $(-2+h)^4$ as $h^4 - 8h^3 + 24h^2 - 32h + 16$ PI
	$= h^4 - 8h^3 + 24h^2 - 31h + 14$	A1F		Seen separately or as part of the gradient expression. Ft one incorrect term in expansion of $(-2+h)^4$
	Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{h^4 - 8h^3 + 24h^2 - 31h + 14 - (14)}{-2 + h - (-2)}$	M1		Use of correct formula for gradient PI
	$= \dfrac{h^4 - 8h^3 + 24h^2 - 31h}{h} =$ $h^3 - 8h^2 + 24h - 31$	A1	5	The four correct terms in any order A0 if incorrect (constant/h) term ignored due printed form of answer
(b)	As $h \to 0$, gradient of line in **(a)** $\to$ gradient of curve at point $(-2, 14)$}	E1		Lim [c's$(p+qh+rh^2+h^3)$] OE $h \to 0$
	{Gradient of curve at point $(-2, 14)$ is} -31	E1	2	NB 'h=0' instead of '$h \to 0$' gets E0 Dependent on previous E1 and printed **form** of answer in (a) obtained convincingly but then ft on c's p value
	Total		**7**	

18.Jan 2013

4	$\int \dfrac{1}{x\sqrt{x}}\,dx = \int x^{-\frac{3}{2}}\,(dx)$	M1		$\int x^{-\frac{3}{2}}$ PI
	$= -2x^{-\frac{1}{2}}\ (+c)$	A1		ACF, can be unsimplified. Condone absence of $+c$
	$-2x^{-\frac{1}{2}} \to 0$ as $x \to \infty$	E1		OE Ft on $k\,x^{-n}$, $n>0$
	$\int_{25}^{\infty} \dfrac{1}{x\sqrt{x}}\,dx = \dfrac{2}{5}$	A1	4	
	Total		**4**	