Practice examination questions for numerical methods

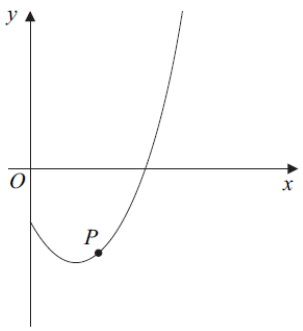
1. Jan 2005

8 [**Figure 2**, printed on the insert, is provided for use in this question.]

The diagram shows a part of the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3 - 2x - 1$$

The point P has coordinates $(1, -2)$.



(a) Taking $x_1 = 1$ as a first approximation to a root of the equation $\mathrm{f}(x) = 0$, use the Newton-Raphson method to find a second approximation, x_2, to the root. *(3 marks)*

(b) On **Figure 2**, draw a straight line to illustrate the Newton-Raphson method as used in part (a).

Mark x_1 and x_2 on **Figure 2**. *(2 marks)*

(c) By considering $\mathrm{f}(2)$, show that the second approximation found in part (a) is not as good as the first approximation. *(2 marks)*

(d) Taking $x_1 = 1.6$ as a first approximation to the root, use the Newton-Raphson method to find a second approximation to the root. Give your answer to three decimal places. *(2 marks)*

2. June 2005

2 A curve satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x$$

where the angle $2x$ is measured in **radians**.

Starting at the point $(0.5, 1)$ on the curve, use a step-by-step method with a step length of 0.1 to estimate the value of y at $x = 0.7$. Give your answer to three significant figures. *(6 marks)*

3. Jan 2006

1 (a) Show that the equation

$$x^3 + 2x - 2 = 0$$

has a root between 0.5 and 1. *(2 marks)*

(b) Use linear interpolation once to find an estimate of this root. Give your answer to two decimal places. *(3 marks)*

4. June 2006

2 A curve satisfies the differential equation

$$\frac{dy}{dx} = \log_{10} x$$

Starting at the point (2, 3) on the curve, use a step-by-step method with a step length of 0.2 to estimate the value of y at $x = 2.4$. Give your answer to three decimal places. *(6 marks)*

5. June 2007

2 (a) Show that the equation

$$x^3 + x - 7 = 0$$

has a root between 1.6 and 1.8. *(3 marks)*

(b) Use interval bisection **twice**, starting with the interval in part (a), to give this root to one decimal place. *(4 marks)*

6. Jan 2008

2 A curve satisfies the differential equation

$$\frac{dy}{dx} = 2^x$$

Starting at the point $(1, 4)$ on the curve, use a step-by-step method with a step length of 0.01 to estimate the value of y at $x = 1.02$. Give your answer to six significant figures. *(5 marks)*

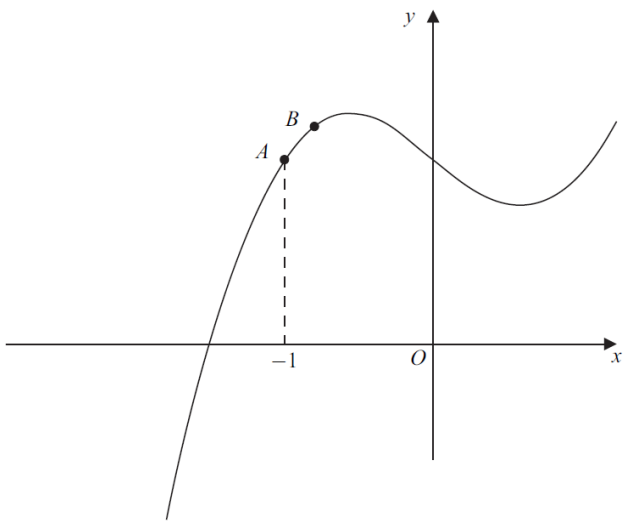
7. Jan 2008

7 [**Figure 1**, printed on the insert, is provided for use in this question.]

The diagram shows the curve

$$y = x^3 - x + 1$$

The points A and B on the curve have x-coordinates -1 and $-1 + h$ respectively.



(a) (i) Show that the y-coordinate of the point B is

$$1 + 2h - 3h^2 + h^3$$ *(3 marks)*

(ii) Find the gradient of the chord AB in the form

$$p + qh + rh^2$$

where p, q and r are integers. *(3 marks)*

(iii) Explain how your answer to part (a)(ii) can be used to find the gradient of the tangent to the curve at A. State the value of this gradient. *(2 marks)*

(b) The equation $x^3 - x + 1 = 0$ has one real root, α.

(i) Taking $x_1 = -1$ as a first approximation to α, use the Newton-Raphson method to find a second approximation, x_2, to α. *(2 marks)*

(ii) On **Figure 1**, draw a straight line to illustrate the Newton-Raphson method as used in part (b)(i). Show the points $(x_2, 0)$ and $(\alpha, 0)$ on your diagram. *(2 marks)*

8. Jan 2009

1 A curve passes through the point $(0, 1)$ and satisfies the differential equation

$$\frac{dy}{dx} = \sqrt{1 + x^2}$$

Starting at the point $(0, 1)$, use a step-by-step method with a step length of 0.2 to estimate the value of y at $x = 0.4$. Give your answer to five decimal places. *(5 marks)*

9. Jan 2010

7 A curve C has equation $y = \dfrac{1}{(x-2)^2}$.

(a) (i) Write down the equations of the asymptotes of the curve C. *(2 marks)*

(ii) Sketch the curve C. *(2 marks)*

(b) The line $y = x - 3$ intersects the curve C at a point which has x-coordinate α.

(i) Show that α lies within the interval $3 < x < 4$. *(2 marks)*

(ii) Starting from the interval $3 < x < 4$, use interval bisection twice to obtain an interval of width 0.25 within which α must lie. *(3 marks)*

10.June 2010

1 A curve passes through the point $(1, 3)$ and satisfies the differential equation

$$\frac{dy}{dx} = 1 + x^3$$

Starting at the point $(1, 3)$, use a step-by-step method with a step length of 0.1 to estimate the y-coordinate of the point on the curve for which $x = 1.3$. Give your answer to three decimal places.

(No credit will be given for methods involving integration.) *(6 marks)*

11.Jan 2011

8 (a) The equation

$$x^3 + 2x^2 + x - 100\,000 = 0$$

has one real root. Taking $x_1 = 50$ as a first approximation to this root, use the Newton-Raphson method to find a second approximation, x_2, to the root. *(3 marks)*

(b) (i) Given that $S_n = \sum_{r=1}^{n} r(3r+1)$, use the formulae for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$S_n = n(n+1)^2$$ *(5 marks)*

(ii) The lowest integer n for which $S_n > 100\,000$ is denoted by N.

Show that

$$N^3 + 2N^2 + N - 100\,000 > 0$$ *(1 mark)*

(c) Find the value of N, justifying your answer. *(3 marks)*

12.June 2011

1 A curve passes through the point $(2, 3)$ and satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{2+x}}$$

Starting at the point $(2, 3)$, use a step-by-step method with a step length of 0.5 to estimate the value of y at $x = 3$. Give your answer to four decimal places.

(5 marks)

13. June 2012

7 The equation

$$24x^3 + 36x^2 + 18x - 5 = 0$$

has one real root, α.

(a) Show that α lies in the interval $0.1 < x < 0.2$. *(2 marks)*

(b) Starting from the interval $0.1 < x < 0.2$, use interval bisection **twice** to obtain an interval of width 0.025 within which α must lie. *(3 marks)*

(c) Taking $x_1 = 0.2$ as a first approximation to α, use the Newton–Raphson method to find a second approximation, x_2, to α. Give your answer to four decimal places. *(4 marks)*

14. Jan 2013

1 A curve passes through the point $(1, 3)$ and satisfies the differential equation

$$\frac{dy}{dx} = \frac{x}{1+x^3}$$

Starting at the point $(1, 3)$, use a step-by-step method with a step length of 0.1 to estimate the value of y at $x = 1.2$. Give your answer to four decimal places.

(5 marks)

Answers

1. Jan 2005

8(a)	$f'(x) = 3x^2 - 2$	B1		
	$x_2 = 1 - \frac{-2}{1} = 3$	M1A1	3	
(b)	Tangent at P drawn	B1		
	x_1 and x_2 shown correctly	B1	2	
(c)	$f(2) = 3 > 0$, so root < 2	E2,1	2	E1 for incomplete explanation
(d)	$x_2 = 1.6 - \frac{-0.104}{5.68} \approx 1.618$	M1A1	2	
	Total		**9**	

2. June 2005

2	$x = 0.5 \Rightarrow y' = \sin 1$	M1		Max 4/6 if degrees used
	First increment is 0.1 sin 1	m1A1		
	$x = 0.6 \Rightarrow y \approx 1.084$	A1		PI by correct answer at end
	Second increment is 0.1 sin 1.2	m1		
	$x = 0.7 \Rightarrow y \approx 1.177 \approx 1.18$	A1F	6	Ft error in $y(0.6)$
	Total		**6**	

3. Jan 2006

Q	Solution	Marks	Totals	Comments
1(a)	f(0.5) = −0.875, f(1) = 1	B1		
	Change of sign, so root between	E1	2	
(b)	Complete line interpolation method	M2,1		M1 for partially correct method
	Estimated root = $\frac{11}{15} \approx 0.73$	A1	3	Allow $\frac{11}{15}$ as answer
	Total		**5**	

4. June 2006

2	1st increment is 0.2 lg 2 ...	M1		or 0.2 lg 2.1 or 0.2 lg 2.2
	... ≈ 0.06021	A1		PI
	$x = 2.2 \Rightarrow y \approx 3.06021$	A1√		PI; ft numerical error
	2nd increment is 0.2 lg 2.2	m1		consistent with first one
	... ≈ 0.06848	A1		PI
	$x = 2.4 \Rightarrow y \approx 3.12869 \approx 3.129$	A1√	6	ft numerical error
	Total		**6**	

5. June 2007

2(a)	f(1.6) = −1.304, f(1.8) = 0.632	B1,B1		Allow 1 dp throughout
	Sign change, so root between	E1	3	
(b)	f(1.7) considered first	M1		
	f(1.7) = −0.387, so root > 1.7	A1		
	f(1.75) = 0.109375, so root ≈ 1.7	m1A1	4	m1 for f(1.65) after error
	Total		**7**	

6. Jan 2008

2	$0.01(2^1)$ added to value of y So $y(1.01) \approx 4.02$ Second increment is $0.01(2^{1.01})$ ... $\approx 0.020\ 139$ So $y(1.02) \approx 4.040\ 14$	M1 A1 m1 A1 A1	 5	Variations possible here PI
	Total		**5**	

7. Jan 2008

7(a)(i)	$(-1+h)^3 = -1 + 3h - 3h^2 + h^3$ $y_B = (-1 + 3h - 3h^2 + h^3) + 1 - h + 1$ ... $= 1 + 2h - 3h^2 + h^3$	B1 B1F B1	 3	PI ft numerical error convincingly shown (AG)
(ii)	Subtraction of 1 and division by h Gradient of chord $= 2 - 3h + h^2$	M1M1 A1	 3	
(iii)	As $h \to 0$, gr(chord) $\to$ gr(tgt) $= 2$	E1B1F	2	E0 if '$h = 0$' used; ft wrong value of p
(b)(i)	$x_2 = -1 - \frac{1}{2} = -1.5$	M1 A1F	 2	 ft wrong gradient
(ii)	Tangent at A drawn α and x_2 shown correctly	M1 A1	 2	 dep't only on the last M1
	Total		**12**	

8. Jan 2009

1	First increment is 0.2, so $y \approx 1.2$ Second increment is $0.2\sqrt{1+0.2^2}$ ... $\approx 0.203\ 961$, so $y \approx 1.403\ 96$	B1B1 M1 A2,1F	 5	PI; variations possible here A1 if accuracy lost; ft num error
	Total		**5**	

9. Jan 2010

7(a)(i)	Asymptotes $x = 2$, $y = 0$	B1B1	2	
(ii)	One correct branch Both branches correct	B1 B1	 2	 no extra branches; $x = 2$ shown
(b)(i)	$f(3) = -1$, $f(4) = 3$ Sign change, so α between 3 and 4	B1 E1	 2	where $f(x) = (x-3)(x-2)^2 - 1$; OE
(ii)	f(3.5) considered first $f(3.5) > 0$ so $3 < \alpha < 3.5$ $f(3.25) < 0$ so $3.25 < \alpha < 3.5$	M1 A1 A1	 3	OE but must consider $x = 3.5$ Some numerical value(s) needed Condone absence of values here
	Total		**9**	

10.June 2010

1	First increment is 0.1×2 (= 0.2) So next value of y is 3.2 Second inc't is $0.1(1 + 1.1^3) = 0.2331$ Third inc't is $0.1(1 + 1.2^3) = 0.2728$ So $y \approx 3.7059 \approx 3.706$	M1 A1 m1A1 A1 A1F	 6	variations possible here PI PI PI ft one numerical error
	Total		**6**	

11.Jan 2011

8(a)	$x_2 = 50 - \dfrac{50^3 + 2(50^2) + 50 - 100\,000}{3(50^2) + 4(50) + 1}$	B1 B1		For numerator (PI by value 30050) For denominator (PI by value 7701)
	$x_2 \approx 46.1$	B1	3	Allow AWRT 46.1
8(b)(i)	$\Sigma r(3r+1) = 3\Sigma r^2 + \Sigma r$	M1		
	$\ldots = 3\left(\frac{1}{6}n\right)(n+1)(2n+1) + \frac{1}{2}n(n+1)$	m1		correct formulae substituted
	$\ldots = \frac{1}{2}n(n+1)(2n+1+1)$	m1m1		m1 for each factor (n and $n+1$)
	$\ldots = n(n+1)^2$ convincingly shown	A1	5	AG
(ii)	Correct expansion of $n(n+1)^2$	B1	1	and conclusion drawn (AG)
(c)	Attempt at value of S_{46} Attempt at value of S_{45} $S_{45} < 100000 < S_{46}$, so $N = 46$	M1 m1 A1	3	
	Alternative method Root of equation in (a) is 45.8 So lowest integer value is 46	(B3)		Allow AWRT 45.7 or 45.8
	Total		**12**	

12.June 2011

1	Attempt at $0.5 \times y'(2)$ (= 0.25)	M1		Other variations are allowed
	$y(2.5) \approx 3.25$	A1		
	$y(3) \approx 3.25 + 0.5\,y'(2.5)$	m1		
	$\approx 3.25 + 0.2357(0)$	A1F		PI; OE; ft c's value for $y(2.5)$
	≈ 3.4857	A1	5	4 dp needed
	Total		**5**	

13.June 2012

7(a)	Let $f(x) = 24x^3 + 36x^2 + 18x - 5$			
	$f(0.1) = -2.816$, $f(0.2) = 0.232$	M1		Both attempted and at least one evaluated correctly to at least 1sf rounded or truncated OE fraction
	Change of sign so α lies between 0.1 and 0.2	A1	2	Need both evaluations correct to above degree of accuracy and 'change of sign OE' <u>and</u> relevant reference to 0.1 and 0.2
(b)	$f(0.15) = -1.409$ (< 0 so root > 0.15)	M1		f(0.15) considered first
	$f(0.175) \approx -0.619$ (< 0 so root > 0.175)	A1		f(0.15) then f(0.175) both evaluated correctly to at least 1sf OE fractions
	α lies between 0.175 and 0.2	A1	3	Dependent on both previous marks gained and no other additional evaluations other than at 0.15 and 0.175
(c)	$f'(x) = 72x^2 + 72x + 18$	B1		PI
	$(x_2 =)$ $0.2 - \dfrac{24(0.2)^3 + 36(0.2)^2 + 18(0.2) - 5}{72(0.2)^2 + 72(0.2) + 18}$	B1 B1		B1 for numerator in correct formula B1 for denominator in correct formula
	= 0.1934 (to 4dp)	B1	4	CAO Must be 0.1934 Do not apply ISW NMS scores 0/4
	Total		**9**	

14.Jan 2013

1	$y_{n+1} \approx y_n + h\,\mathrm{f}(x_n)$			OE
	$h\,y'(1) = 0.1 \times y'(1)$ (=0.05)	M1		Attempt to find $h\,y'(1)$. PI by eg 3.05 for $y(1.1)$
	$y(1.1) \approx 3 + 0.05 = 3.05$	A1		
	$y(1.2) \approx y(1.1) + 0.1 \times y'(1.1) = 3.05 + 0.1 \times y'(1.1)$	m1		Attempt to find $y(1+0.1)+0.1\times y'(1+0.1)$ must see evidence of calculation if correct ft [0.047..+c's $y(1.1)$] value not obtained
	$\approx 3.05 + 0.1 \times \dfrac{1.1}{1+1.1^3} \quad \left(= 3.05 + 0.1 \times \dfrac{1100}{2331}\right)$			
	$\approx 3.05 + 0.047(19\ldots..)$	A1F		OE; ft on [0.047..+c's $y(1.1)$] value; PI
	≈ 3.0972 (to 4 d.p.)	A1	5	Must be 4 dp.
	Total		**5**	