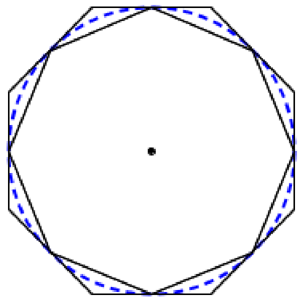
$\boldsymbol{Polygon\ \pi}$

Archimedes, the famous Greek mathematician, used the perimeter of polygons as an approximation for circumference to calculate increasingly good estimates of the constant π:

- Construct a regular polygon.
- By joining the midpoints of each side, construct a smaller one inside it.
- Find the perimeter of both polygons, and calculate the average.
- Divide by the diameter of the circle that fits between the two polygons.

This works because it is always possible to draw a circle which fits inside the larger polygon and outside the smaller. The circumference of the circle must therefore be greater than the perimeter of the smaller polygon, but less than the perimeter of the larger.



Archimedes did not have a calculator, and didn't use trigonometry as we currently understand it, so his methods involved the use of Pythagoras' Theorem, and relied on fractional approximations for surds $\left(\sqrt{3} \approx \frac{1351}{780}\right)$. Considering this, he did very well, using a nifty doubling method to go from 6 sides to 12, then 24, 48 and finally 96 sides, proving: $\frac{223}{71} < \pi < \frac{22}{7}$

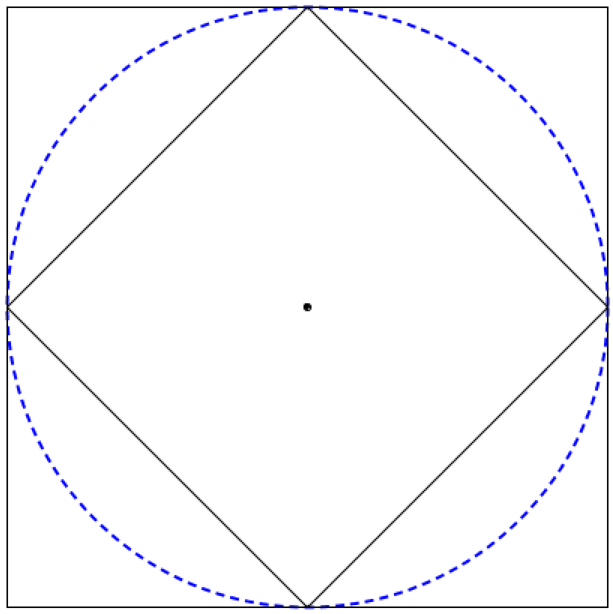
(taking an average yields an estimate which is correct to 3 decimal places: $\pi_{est} = 3.14185110663984$ ...)

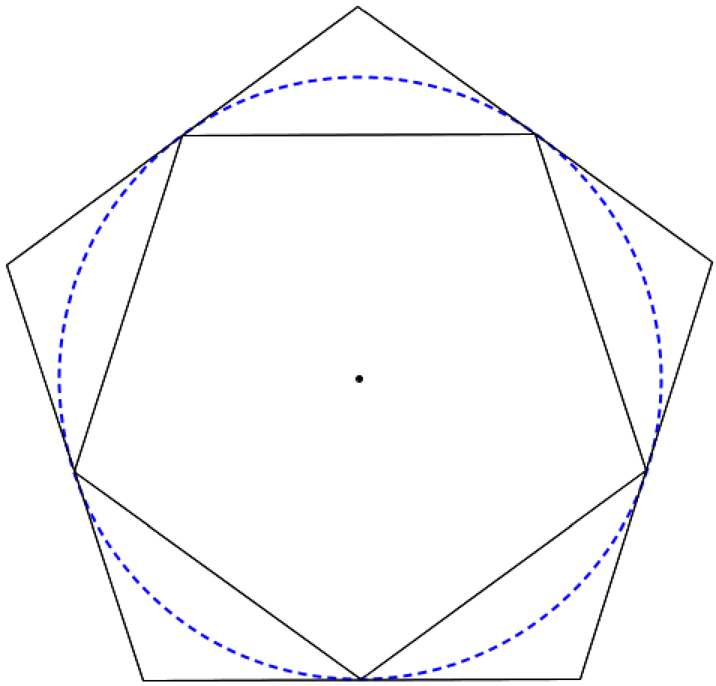
$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651 3 \ldots$

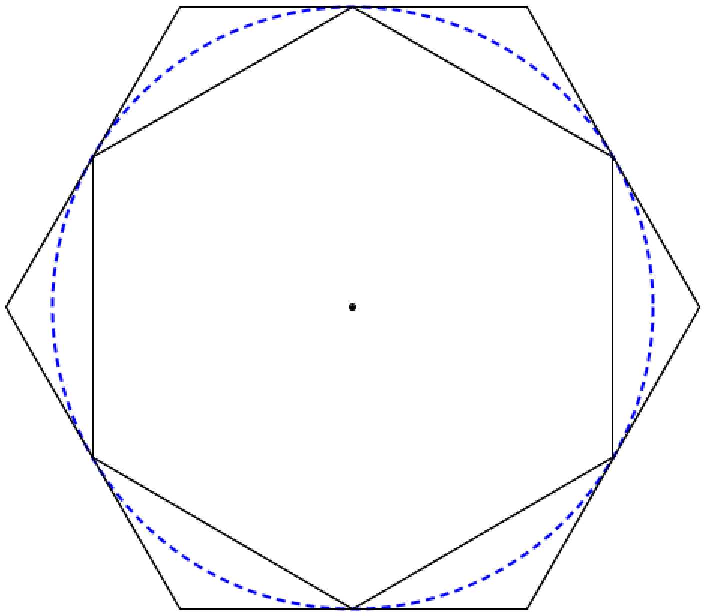
Estimation Using Trigonometry

Apply this method to a square, pentagon, hexagon, and finally a hendecagon (100 sides).

1. Small Polygon Triangle: Make a triangle between the midpoint of a side, the corner beside it and the centre of the circle. Label the angle at the centre and the side which is the radius of the circle (recall we are using a radius of 1 throughout this process).	
2. Perimeter of Small Polygon: Use right-angled trigonometry to determine the length of the half-side, and multiply up to determine the perimeter of the whole small polygon.	
3. Large Polygon Triangle: Make a triangle between the midpoint of a side, the corner beside it and the centre of the circle. Label the angle at the centre and the side which is the radius of the circle (note that this was the hypotenuse of the smaller triangle).	
4. Perimeter of Large Polygon: Use right-angled trigonometry to determine the length of the half-side, and multiply up to determine the perimeter of the whole large polygon.	
5. Estimate for π: Find the average of the two perimeters, and divide by the diameter (2) to determine an estimate for π.	



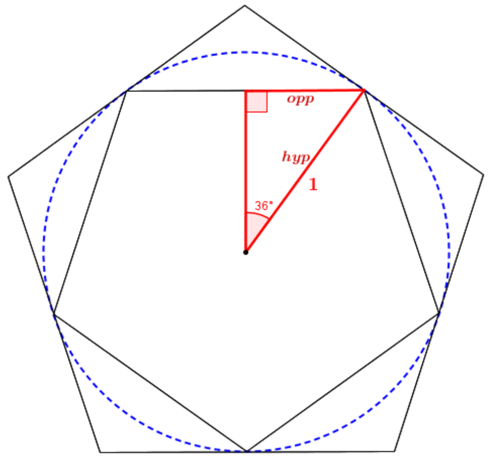




Estimation Using Trigonometry SOLUTIONS

1. Small Polygon Triangle:

Make a triangle between the midpoint of a side, the corner beside it and the centre of the circle.
Label the angle at the centre and the side which is the radius of the circle (recall we are using a radius of 1 throughout this process).



2. Perimeter of Small Polygon:

Use right-angled trigonometry to determine the length of the half-side, and multiply up to determine the perimeter of the whole small polygon.

$\sin\theta = \frac{opp}{hyp} \implies \sin 36 = \frac{x}{1}$

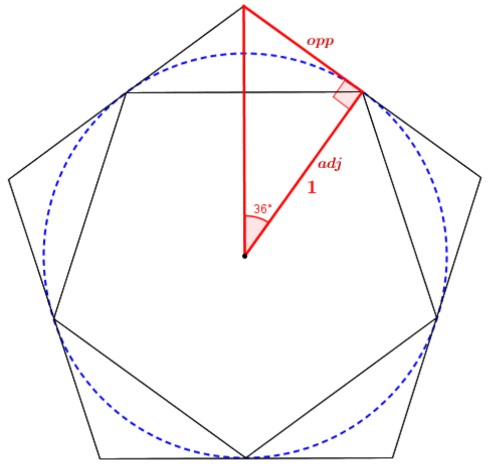
$x = 0.5877\ldots$

$P = 2 \times 5 \times 0.5877\ldots$

$P_{small} = \mathbf{5.877}\ldots$

3. Large Polygon Triangle:

Make a triangle between the midpoint of a side, the corner beside it and the centre of the circle.
Label the angle at the centre and the side which is the radius of the circle (note that this was the hypotenuse of the smaller triangle).



4. Perimeter of Large Polygon:

Use right-angled trigonometry to determine the length of the half-side, and multiply up to determine the perimeter of the whole large polygon.

$\tan\theta = \frac{opp}{adj} \implies \tan 36 = \frac{x}{1}$

$x = 0.7265\ldots$

$P = 2 \times 5 \times 0.7265\ldots$

$P_{large} = \mathbf{7.265}\ldots$

5. Estimate for π:

Find the average of the two perimeters, and divide by the diameter (2) to determine an estimate for π.

$$P_{average} = \frac{5.877\ldots + 7.265\ldots}{2}$$

$$\pi_{5-sides\ estimate} = 3.285819\ldots$$

Estimates for n sided shapes given below:

n	4	5	6	7	8	…	100	…	1000
π_{est}	3.414214	3.285819	3.232051	3.204104	3.187588	…	3.141851	…	3.141595