Introduction to Polar Coordinates in Mechanics (for AQA Mechanics 5)

Until now, we have dealt with displacement, velocity and acceleration in Cartesian coordinates - that is, in relation to fixed perpendicular directions defined by the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Consider this exam question to be reminded how well this system works for circular motion:

AQA Mechanics 2B, Jun '12

4 A particle moves on a horizontal plane, in which the unit vectors **i** and **j** are perpendicular.

At time t, the particle's position vector, **r**, is given by

$$\mathbf{r} = 4\cos 3t\mathbf{i} - 4\sin 3t\mathbf{j}$$

(a) Prove that the particle is moving on a circle, which has its centre at the origin.

(2 marks)

Note that the position vector is given in $\boldsymbol{i}, \boldsymbol{j}$ form, with each component in terms of the time t.

To prove that the particle is performing circular motion about the origin, it is sufficient to show that the *distance* from the origin is constant. Since $\boldsymbol{r}$ is the displacement vector, the distance is given by $|\boldsymbol{r}|$:

$$|\boldsymbol{r}| = \sqrt{(4\cos 3t)^2 + (-4\sin 3t)^2} = 4\sqrt{\cos^2 3t + \sin^2 3t} = 4$$

$$Constant \quad \Longrightarrow \quad Exhibiting\ circular\ motion\ with\ centre\ the\ origin.$$

By considering how the $\boldsymbol{i}$ and $\boldsymbol{j}$ components vary as $3t$ varies between 0 and 2π, we can get an impression not just of position, but also of velocity. Eg, while $0 \leq 3t \leq \frac{\pi}{2}$ the particle is in the fourth quadrant – in the $\boldsymbol{i}$ direction the position is positive but decreasing slowly, and in the $\boldsymbol{j}$ direction the position is negative and decreasing rapidly. In other words, the particle is travelling clockwise around a circle of radius 4 about the origin, starting from the point $4\boldsymbol{i} + 0\boldsymbol{j}$.

(b) Find an expression for the velocity of the particle at time t. *(2 marks)*

Recall that the velocity can be found from displacement using the result $\boldsymbol{v} = \frac{dr}{dt}$, and when $\boldsymbol{v}$ and $\boldsymbol{r}$ are vectors, this means differentiating the $\boldsymbol{i}$ and $\boldsymbol{j}$ components separately, with respect to time (these components are perpendicular, so do not directly affect one another).

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -12\sin 3t\,\boldsymbol{i} - 12\cos 3t\,\boldsymbol{j}$$

By considering the components of velocity at various points, just as we did for displacement, we can get an idea of the velocity of the particle. For instance, when $\frac{\pi}{2} \leq 3t \leq \pi$, the particle is moving left (quickly at first, but with decreasing speed) and up (slowly at first, but with increasing speed). Note also that, using the same method as in part a), we could use Pythagoras to show that the magnitude of velocity (the speed) is constant. Therefore while direction is constantly changing, the particle is travelling at a constant speed. Since the acceleration of the particle is affecting both components of velocity, it is harder to see how this is changing directly from this expression, but it will become clearer once we find the acceleration vector. Note, however, that as the $\boldsymbol{i}$ component of velocity increases, the $\boldsymbol{j}$ component decreases and vice versa. We should find that the acceleration of the particle is of constant magnitude even though its direction is constantly changing.

(c) Find an expression for the acceleration of the particle at time t. *(2 marks)*

Recall that the acceleration vector can be found from velocity using the result $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$.

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -36\cos 3t\,\boldsymbol{i} + 36\sin 3t\,\boldsymbol{j}$$

Note firstly that it would be easy to show, as in part a), that acceleration is constant in magnitude (if not direction). Also note the similarity of this expression to that of displacement. The only difference is the signs (negative $\boldsymbol{i}$ and positive $\boldsymbol{j}$ in this case) and the magnitude. So the acceleration is clearly directly related to the displacement of the particle. This is a key feature of circular motion.

(d) The acceleration of the particle can be written as

$$\mathbf{a} = k\mathbf{r}$$

where k is a constant.

Find the value of k. *(2 marks)*

This is a straightforward result to prove, but it is crucial to the idea of circular motion.

$$\boldsymbol{a} = -36\cos 3t\,\boldsymbol{i} + 36\sin 3t\,\boldsymbol{j} = -9(4\cos 3t\,\boldsymbol{i} - 4\sin 3t\,\boldsymbol{j}) = -9\boldsymbol{r}$$

Since the acceleration vector is a scalar multiple of the displacement vector, the two vectors are parallel. This means acceleration *always* acts along the same line as the displacement of the particle from the origin (that is, along the radius of the circular motion).

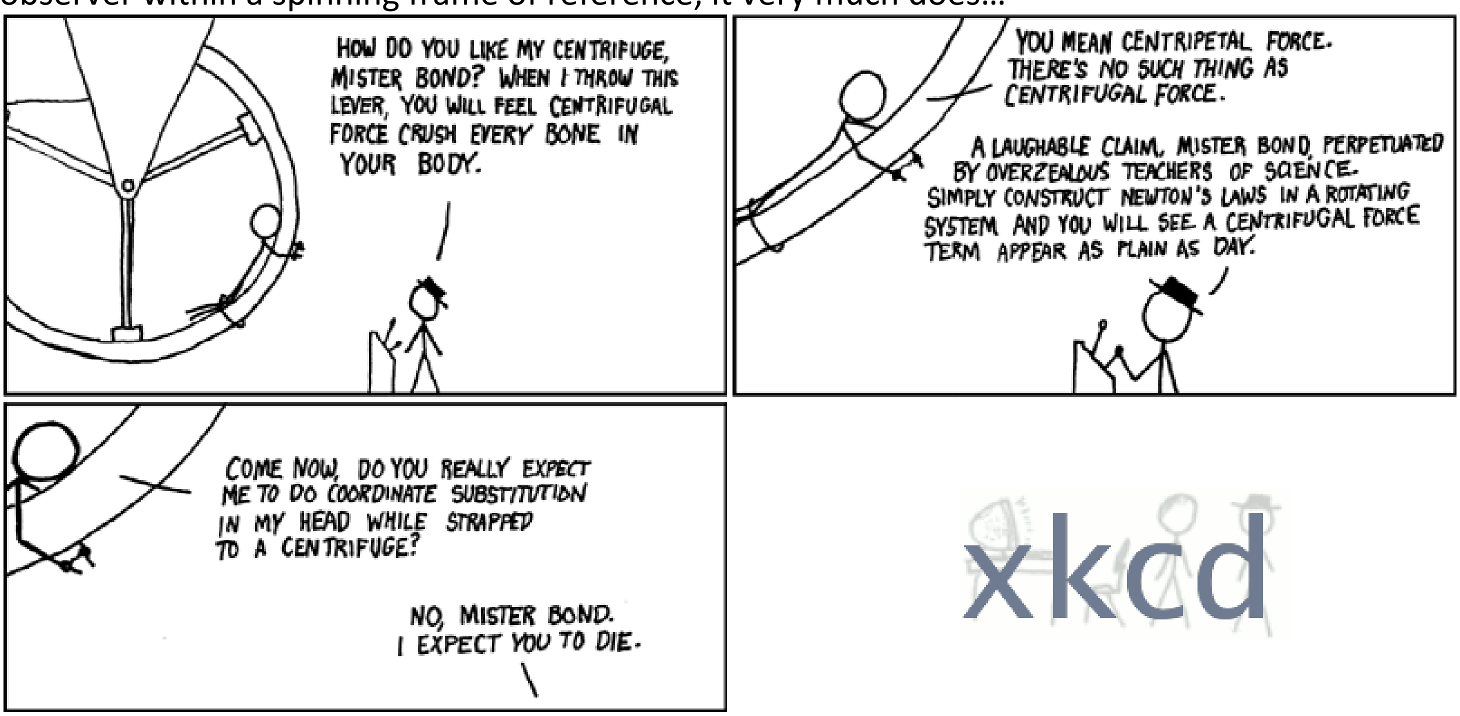
(e) State the direction of the acceleration of the particle. *(1 mark)*

$$Towards\ the\ centre\ of\ the\ circle\ (the\ origin)$$

Note the negative sign – this tells us that the acceleration acts in the opposite direction to displacement. Displacement measures the position of the particle relative to the origin, and so it always points from the origin to the particle. Therefore acceleration points from the particle to the origin. This fits in with the whole concept of centripetal force and centripetal acceleration. Since the velocity is constantly changing direction, the acceleration vector must be constantly changing direction, and since acceleration is always pointing radially (towards the centre), it cannot affect the magnitude of velocity which always points tangentially (at right angles to the radial direction).

Why Polar Coordinates?

The motion described in the question above required the use of the trigonometric functions $sine$ and $cosine$. It was possible to describe motion in the Cartesian coordinate system, but was somewhat clumsy. Since the directions we are really interested in with this sort of motion constantly change, it makes sense to have the unit vectors we relate to changing in the same way. Using a different coordinate system like this often simplifies what would otherwise be intractable problems. Circular motion itself is not too bad, but other forms of curvilinear motion (such as motion in a spiral) can get rather complicated. In addition, phenomena such as centrifugal force (the apparent force experienced by an object moving in a curve) can be more easily grasped. While this force may not appear to exist 'in reality', from the point of view of an observer within a spinning frame of reference, it very much does…



Constructing the Polar Coordinate System

A note on notation

To stop an already complicated system of formulae looking completely ludicrous, we will use the accepted notation $\frac{dx}{dt} = \dot{x}$ and $\frac{d^2x}{dt^2} = \ddot{x}$. These are the derivative and second derivative of x (which is a function of t) with respect to t.
For instance, if $x = 3\cos t$ then $\dot{x} = -3\sin t$ and $\ddot{x} = -3\cos t$.

Implications of implicit differentiation

Recall that if we need to differentiate an expression in terms of one variable with respect to another, the chain rule allows us to write: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.
Combining this idea with the notation above, we can say things like: $\frac{d(r^2)}{dt} = 2r\dot{r}$.
Note that, like any letters used here which aren't vectors, r is a function of time.
When a letter represents a vector, it will be written in **bold**.

The Cartesian system

We begin by considering what we already know – the Cartesian system:

The general form of displacement would be given as: $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$
where x and y are functions of time, and $\boldsymbol{i}$ and $\boldsymbol{j}$ are fixed perpendicular unit vectors in the plane.

The velocity vector can found from displacement: $\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$
Since velocity is the rate of change of displacement, it can be described precisely, in vector form, by considering the rate of change separately in each of the two *fixed* perpendicular directions.

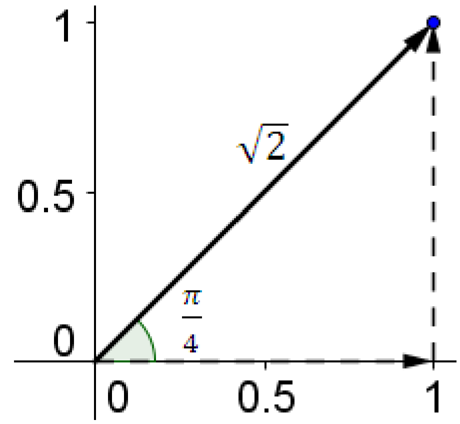
Similarly, the acceleration vector can be found: $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$

Note the use of $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ for the velocity and acceleration vectors. The results above are the definitions of these vectors, and their derivatives cannot be found directly, but depend on these definitions. This will become important – and less intuitive – when we define equivalent expressions for displacement ($\boldsymbol{r}$), velocity ($\dot{\boldsymbol{r}}$) and acceleration ($\ddot{\boldsymbol{r}}$) in polar coordinates.

The concept of Polar Coordinates

Essentially, polar coordinates boil down to a different way of defining position in the plane. Consider the point $(1,1)$ in Cartesian coordinates. It can be thought of as 1 unit along (to the right) and 1 unit up.

However, it could also be identified by its distance from the origin $\left(\sqrt{2}\right)$ and the anticlockwise angle its position vector makes with the positive x axis $\left(\frac{\pi}{4}\right)$:

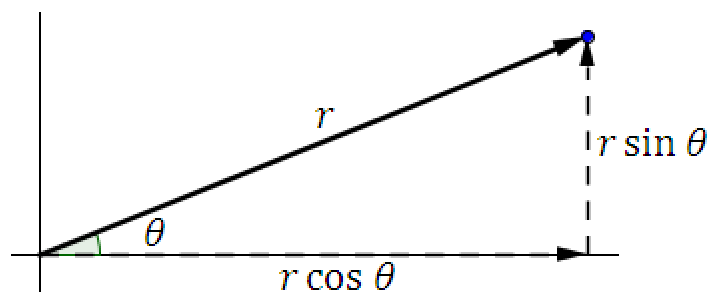


Instead of (x, y), we have (r, θ), where r is the magnitude of the position vector (the distance from the origin) and θ is the angle. Our point $(1,1)$ in Cartesian coordinates becomes $\left(\sqrt{2}, \frac{\pi}{4}\right)$ in polar form.

It may seem like a lot of effort to go to just for a different way of describing something we already have a system for, but certain functions naturally lend themselves to being described in this way. The unit circle, for instance: Cartesian form: $x^2 + y^2 = 1$; Polar form: $r = 1$.

Making the link

By considering the vector triangle opposite, note that the x coordinate is equivalent to $r\cos\theta$ and the y coordinate to $r\sin\theta$. This allows us to make a link between the system we are already familiar with (Cartesian) and the one we are trying to develop.



Transverse and Radial components

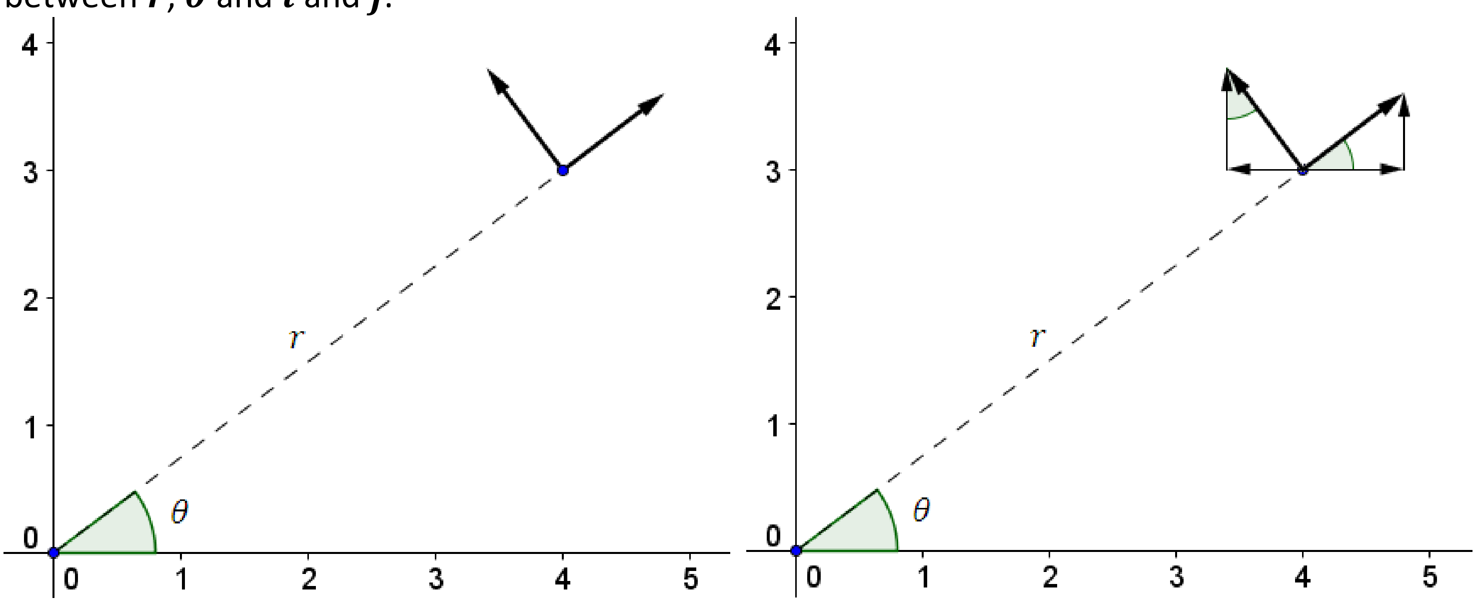
While the magnitude and angle system is important to understand, in order to describe *motion* in polar form it is necessary to define perpendicular unit vectors. The vector $\hat{\boldsymbol{r}}$ is a unit vector in the radial (from the centre) direction, and $\hat{\boldsymbol{\theta}}$ is the unit vector in the transverse direction (at right angles to $\hat{\boldsymbol{r}}$).

The *radial unit vector* $\hat{\boldsymbol{r}}$ always points one unit in the same direction as the position vector of the point in question (in the direction of increasing radius r).

The *transverse unit vector* $\hat{\boldsymbol{\theta}}$ points one unit at right angles to this vector (in the direction of increasing angle θ).

Note: The $\hat{\boldsymbol{r}}$ ('r-hat') notation indicates a unit vector. I am not using it for $\boldsymbol{i}$ and $\boldsymbol{j}$ simply because there is nothing to confuse them with, whereas by the time we're finished there will be r, $\dot{r}$, $\ddot{r}$, $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ to keep track of in addition to the unit vector $\hat{\boldsymbol{r}}$.

By considering an arbitrary point in the plane, we can finally start to identify the relationships between $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{i}$ and $\boldsymbol{j}$.



The vector pointing directly away from the origin is the unit vector $\hat{\boldsymbol{r}}$, and the unit vector at right angles to the radial direction is the transverse unit vector $\hat{\boldsymbol{\theta}}$.

Note that the marked angles in the right hand diagram are both θ, and since the unit vectors have length 1, the component vectors in the $\boldsymbol{i}$ and $\boldsymbol{j}$ directions are:

$$\hat{\boldsymbol{r}} = cos\,\theta\,\boldsymbol{i} + sin\,\theta\,\boldsymbol{j} \qquad and \qquad \hat{\boldsymbol{\theta}} = -\,sin\,\theta\,\boldsymbol{i} + cos\,\theta\,\boldsymbol{j}$$

Displacement in terms of transverse and radial unit vectors

This is the most intuitive of the results that follow, and is easily derived from the Cartesian definitions:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$

This can be derived directly from the definition of displacement in the Cartesian coordinates system, and the link between x and y in that system with r and θ:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j} = r(cos\,\theta\,\boldsymbol{i} + sin\,\theta\,\boldsymbol{j}) = r\hat{\boldsymbol{r}}$$

Since the displacement is simply the position of the point in the plane, we can locate it by simply going the right distance in the right direction. The variable r (a function of time) tells us how far away the particle is from the origin at any given moment, and the radial unit vector is already defined in terms of the direction of the particle from the origin, so we can simply multiply the required distance by the unit vector we already have which points in the required direction. Note that there is no transverse component to this at all – this is not an accident; the unit vector $\hat{\boldsymbol{r}}$ was defined specifically to always point in the direction of displacement, so the transverse unit vector is not needed.

Velocity in terms of transverse and radial unit vectors

This is somewhat less clear than the previous result, but through the careful application of some basic differentiation techniques, and using the notation explained earlier, we get:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

Starting with the Cartesian form, and using product rule and chain rule (implicit differentiation), we can derive an expression for $\boldsymbol{v}$, or $\dot{\boldsymbol{r}}$ as it is more often referred to.

$$x = r\cos\theta \quad \Longrightarrow \quad \dot{x} = \dot{r}\cos\theta + r\left(-\sin\theta\,\dot{\theta}\right) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$y = r\sin\theta \quad \Longrightarrow \quad \dot{y} = \dot{r}\sin\theta + r\left(\cos\theta\,\dot{\theta}\right) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\boldsymbol{i} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\boldsymbol{j}$$

$$= \dot{r}(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}) + r\dot{\theta}(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}) = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

Note that r and θ are simply functions of time, hence the need to use chain rule when differentiating with respect to time.

We designed our coordinate system with displacement in mind, which is why the displacement vector is so straightforward (there is never any transverse component). However, this means that the form we arrive at for velocity takes a little explaining. Fortunately, when broken down, it is not a huge step from the idea of velocity in circular motion.
Note first of all that the radial component of velocity is simply $\dot{r}$ – the rate of change of the radial distance. If the particle is moving directly away from the centre (and not changing its angle at all), this is the only component of velocity it would have.
Secondly, in order to understand the transverse component we should consider what $\dot{\theta}$ represents – it is the rate of change of the angle, θ, with respect to time. In other words, the angular velocity of the particle, which is often written as ω.

When measuring the overall velocity of a particle, we need to take into account its velocity in both directions (radial and tangential/transverse). The radial velocity is the rate at which the radius is changing, $\dot{r}$, and the transverse speed must be the rate at which it is moving at right angles to the radius. Just as tangential speed in circular motion is given by $v = r\omega$, so our tangential speed is also $r\omega$, or, using our notation, $r\dot{\theta}$.

When the two components of velocity are combined, and written along with the radial and transverse unit vectors, we get:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

Angular Velocity side note:
Since the angular velocity measures radians turned through per second, it takes no account of the size of the circle. So the angular velocity of the Earth around the sun is microscopic because it takes a whole year just to complete a single turn (2π radians in 31.5 million seconds, or $2 \times 10^{-7}\ rad\ s^{-1}$), while the angular velocity of a PowerBall gyroscope might easily be 10,000 rpm ($2\pi \times 10000$ radians in 60 seconds, or $1000\ rad\ s^{-1}$).
Since a turn of one radian represents an arc length of exactly one radius-length, multiplying the angular velocity by the radius gives the actual speed (in ms^{-1}).

Acceleration in terms of transverse and radial unit vectors

You might be tempted to differentiate the radial and transverse components directly, using the expression above for velocity, but this is not in line with our accepted definition of acceleration. Since the unit vectors we are measuring with respect to are changing as time goes by, we need to take that into account. So, again, our result comes from examining the Cartesian form:

$$\ddot{\boldsymbol{r}} = \boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

The trick with these proofs is to work out the $\boldsymbol{i}$ component and $\boldsymbol{j}$ component separately, then combine and take out combinations of $sine$ and $cosine$ which correspond to the definitions of our new unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\Rightarrow \quad \ddot{x} = \left(\ddot{r}\cos\theta + \dot{r}\left(-\sin\theta\,\dot{\theta}\right)\right) - \left(\dot{r}\dot{\theta}\sin\theta + r\ddot{\theta}\sin\theta + r\dot{\theta}\cos\theta\,\dot{\theta}\right)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\Rightarrow \quad \ddot{y} = \left(\ddot{r}\sin\theta + \dot{r}\cos\theta\,\dot{\theta}\right) + \left(\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}\sin\theta\,\dot{\theta}\right)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\cos\theta$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} = \left(\ddot{r} - r\dot{\theta}^2\right)(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}) + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j})$$

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

The acceleration expression is the most complicated-looking of the three, but looking at the individual components in turn will enable us to identify the key components and how they fit in.

Firstly, the radial component is the combined effect of the motion along the radial line and centripetal acceleration. Note that $r\dot{\theta}^2 = r\omega^2$ which should be familiar from circular motion (also note that the term is negative since the centripetal force must act towards the origin).

Secondly, the transverse component is a combination of $r\ddot{\theta}$ and $2\dot{r}\dot{\theta}$. Taking the $r\ddot{\theta}$ term first, we should recognize that $\ddot{\theta}$ is the rate of change of angular velocity (since $\dot{\theta} = \omega$), so it makes sense that it should form part of the transverse acceleration – if the velocity in the transverse direction is changing, the acceleration in that direction will depend on the rate of change of the transverse velocity. This effect is magnified if the particle is further from the origin – that is, when r is large. So by multiplying by r we get a term which provides part of the total transverse acceleration. Next, to understand the $2\dot{r}\dot{\theta}$ term, we need to recognize that it comes about as a consequence of a change in transverse velocity. The previous term is simply the acceleration required at a given distance from the origin to change angular speed, but this term comes into effect when the radius (the distance from the origin) is also changing. It is directly related to the angular velocity, hence the $\dot{\theta}$ term, but since it depends on the rate at which the radius changes, it must also contain $\dot{r}$.

Some useful case studies

Motion	***Velocity***	***Acceleration***
In general:	$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$	$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$
Motion at a constant speed in a straight line from the origin.	$\dot{r}\hat{\boldsymbol{r}}$ Since θ is constant, $\dot{\theta} = 0$. $\dot{r}$ will be constant.	0 No radial acceleration since $\dot{r}$ is constant, so $\ddot{r} = 0$, and $\dot{\theta} = 0$. No transverse acceleration since $\dot{\theta} = \ddot{\theta} = 0$.
Motion at a constant speed in a circle.	$r\dot{\theta}\hat{\boldsymbol{\theta}}$ $\dot{r} = 0$ since r is constant – there is no radial velocity. Transverse velocity is $r\dot{\theta}$ or $r\omega$. Since the speed around the circle is constant, $\dot{\theta}$ is constant.	$-r\dot{\theta}^2\hat{\boldsymbol{r}}$ $\dot{r} = \ddot{r} = 0$ since r is constant. And since $\dot{\theta}$ is constant, $\ddot{\theta} = 0$. There is no transverse acceleration, and the only component of radial acceleration is what is known as centripetal acceleration; $r\omega^2$ acting towards the centre of the circle.
Varied motion along a fixed radial line.	$\dot{r}\hat{\boldsymbol{r}}$ Since θ is constant, $\dot{\theta} = 0$, so there is no transverse component of velocity.	$\ddot{r}\hat{\boldsymbol{r}}$ $\dot{\theta} = \ddot{\theta} = 0$ so there is no centripetal component of radial acceleration, and no transverse acceleration.
Motion around a fixed circle with variable angular velocity.	$r\dot{\theta}\hat{\boldsymbol{\theta}}$ $\dot{r} = 0$ since r is constant, so there is no radial component of velocity. The transverse component of velocity varies as $\dot{\theta}$ (or ω) varies, but is proportional to the fixed radius r.	$-r\dot{\theta}^2\hat{\boldsymbol{r}} + r\ddot{\theta}\hat{\boldsymbol{\theta}}$ $\dot{r} = \ddot{r} = 0$ since r is constant. The only radial component is the centripetal force component causing the circular motion, and the transverse component – since the radius is constant – is just the rate of change of the angular velocity for this fixed radius.
Variable motion along a radial line rotating at a constant rate.	$\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ $\dot{\theta}$ is constant.	$\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}$ $\ddot{\theta} = 0$ since $\dot{\theta}$ is constant. The radial component is still composed of two parts, since the object is moving back and forth along the line and since centripetal acceleration is required for the transverse velocity to be constantly changing direction (curved motion). Since $\ddot{\theta} = 0$, the only transverse component is that which depends on both angular velocity and the rate of change of the radius.

Final notes

Angular momentum is an analogous concept to linear momentum. In linear momentum, the mass and the velocity are the key components, but with angular momentum, an additional component of the radius is involved. Like linear momentum, angular momentum is conserved, which is why an ice-skater spins faster when they pull their legs and arms in to reduce the radius.

If we multiply the radius by the angular velocity we get $r^2\dot{\theta}$. Unless a transverse force is applied to change angular momentum, this will be constant. Since transverse force is directly linked to transverse acceleration, the result $r^2\dot{\theta} = k$, where k is a constant, is equivalent to saying that the transverse component of acceleration is 0.

Proof:

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = r\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$r^2\dot{\theta} = k \quad where\ k\ is\ a\ constant \quad \Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Example question

AQA Mechanics 5, Jun '12

5 A particle P, of mass m, is attached to one end of a light elastic string. The other end of the string is attached to a fixed point O on a smooth horizontal table. The string is of natural length a and modulus of elasticity $2mg$.

The string is extended to a length $3a$, and P is projected on the table at right angles to the string with speed U, as shown in the diagram.

During the subsequent motion, the polar coordinates of P with respect to O are (r, θ).

(a) (i) Explain why the transverse component of the acceleration of P is zero. *(1 mark)*

The only resultant force acting on the particle is tension in the string (although it is moving transversely, it has no transverse component of acceleration since there is no force acting at right angles to the string). Since the only force is radial, the transverse component of acceleration must be zero.

(ii) Find, in terms of a and U, an expression for $r^2\dot{\theta}$. *(2 marks)*

$r^2\dot{\theta}$ is the angular momentum term, which must be constant since there is no transverse component of acceleration.

$$r^2\dot{\theta} = r(r\dot{\theta})$$

At the start of motion, $r = 3a$ and $r\dot{\theta} = U$ (since $\dot{\theta}$ is angular velocity, so $r\dot{\theta}$ is the transverse component of velocity). Therefore, since $r^2\dot{\theta}$ is constant throughout the motion, the value at the start must be the same throughout, and is equal to:

$$r^2\dot{\theta} = r(r\dot{\theta}) = 3aU$$

(b) During the motion, the maximum value of r is $4a$.

(i) Show that the speed of P in the position when r is at its maximum is $\frac{3U}{4}$. *(3 marks)*

This comes back to the angular momentum, $r^2\dot{\theta}$, which has been found to be equal to $3aU$ throughout the motion. At the maximum value of r, since the angular momentum is constant:

$$3aU = 4aV \quad \Rightarrow \quad V = \frac{3U}{4}$$

(ii) Find an expression, in terms of a, g, m and U, for the total energy of the system at the moment of projection. *(3 marks)*

We have the initial speed, and enough information about the elastic string to find the total energy of the system, which will be a combination of kinetic and elastic potential:

$$Initial\ Energy = \frac{1}{2}mv^2 + \frac{\lambda e^2}{2l} = \frac{1}{2}mU^2 + \frac{2mg(2a)^2}{2a} = \frac{1}{2}mU^2 + 4mga$$

(iii) Hence find, in terms of a and g, the value of U. *(4 marks)*

At the point when r is maximal:

$$Total\ Energy = \frac{1}{2}m\left(\frac{3U}{4}\right)^2 + \frac{2mg(3a)^2}{2a} = \frac{9}{32}mU^2 + 9mga$$

Since energy is conserved:

$$\frac{1}{2}mU^2 + 4mga = \frac{9}{32}mU^2 + 9mga$$

$$\Rightarrow \quad \frac{7mU^2}{32} = 5mga$$

$$\Rightarrow \quad U = \sqrt{\frac{160ga}{7}}$$

(iv) When r is at its maximum, find the magnitude and direction of the acceleration of P. *(3 marks)*

Using Hooke's law, at maximum stretch the tension will be $T = \frac{\lambda x}{l} = \frac{2mg(3a)}{a} = 6mg$

Since this is the resultant force acting on the particle, we can use $F = ma$ to find the radial component of acceleration (and, as we have already established, there is no transverse component). $\ddot{r} = \frac{F}{m} = 6g$. Note that this is pulling towards the centre, making the direction of acceleration negative in relation to the radial unit vector.

Additional note:
Since, as we have already established, the transverse component of acceleration is zero, the direction must be purely radial.

Radial acceleration is given by:

$$\ddot{r} - r\dot{\theta}^2$$

But since, at maximum r, $\dot{r} = 0$, $\ddot{r} = 0$, so this is simply $-r\dot{\theta}^2$. Since, using force and acceleration, we have already found this value, we could use our result to work backwards and find the angular velocity at this point:

$$-a\dot{\theta}^2 = -6ga \quad \Rightarrow \quad \dot{\theta}^2 = 6g \quad \Rightarrow \quad \dot{\theta} = \sqrt{6g}$$