Pascal's Triangle Investigation

The basics

Definition: **Pascal's Triangle** is a lattice of numbers where each number is the sum of the two directly above it. It starts with a single 1, and the first few rows look like this:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots
\end{array}
$$

Some background

Although named after 17th century mathematician Blaise Pascal, this triangle – built from such a simple set of rules – has been found in the mathematical writings of other cultures from many hundreds of years ago. Chinese mathematicians used it as a calculating device, for generating square and cube roots.

The investigation

The aim is to look for number patterns and recognisable sequences within the triangle. Some are obvious and trivial, while others are highly complex.

Getting started

- Construct the first few rows of Pascal's Triangle to become familiar with the process.
- There is a copy of the first 18 rows (row 0 to row 17) on the back for you to use.
- Identify any patterns/symmetries you notice straight away (even if they seem obvious).
- Now let your imagination run wild. You can follow horizontal, vertical or diagonal lines, you can skip numbers. You can look for sequences, add sets of numbers, or find the difference between them. You can look for interesting patterns by highlighting numbers with particular properties (share a factor, are square, are prime, etc).
- Try to justify or prove any results you notice. Algebra will be useful for this.

Challenges

- How much can you tell me about the numbers of the 100th row of Pascal's Triangle?
- Multiply out the brackets in the expression $(x+1)^{10}$.
- Four people are to be selected at random from a class of 12 to compete in a challenge. How many different four-person teams are possible?

Extension

Try starting a triangle with the same row-by-row rules, but with $1 \quad 2$ on the second row instead of $1 \quad 1$. What patterns remain the same? What changes?

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1

Pascal's Triangle Investigation SOLUTIONS

Disclaimer: there are *loads* of patterns and results to be found in Pascal's triangle.
Here I list just a few. For more ideas, or to check a conjecture, try searching online.

For the purposes of these rules, I am numbering rows starting from 0, so that 'row 1' refers to the second line (1 1), not the very top of the triangle.

A few interesting patterns:

- The triangle is symmetrical about the central vertical line.
- The n^{th} row always starts with 1, and the next number (where it exists) is always n.
- The n^{th} row adds up to 2^n. In fact, rows are often numbered from 0.
  - *Proof: If the elements of one row are $a_1, a_2, a_3, \dots a_k$ then the elements of the next row will be $a_1, a_1 + a_2, a_2 + a_3, \dots, a_{k-2} + a_{k-1}, a_{k-1} + a_k, a_k$, which, since it contains every element of the first row exactly twice, will have double the sum.*
- The 'zeroth' diagonal (along either the left or right edge) is the constant sequence $T(n) = 1$.
- The first diagonal is the linear sequence $T(n) = n$.
- The second diagonal is the quadratic sequence $T(n) = \frac{n}{2}(n+1)$ (triangular numbers).
- The third diagonal is the cubic sequence $T(n) = \frac{n}{3!}(n+1)(n+2)$ (tetrahedral numbers).
- The fourth diagonal is the quartic sequence $T(n) = \frac{n}{4!}(n+1)(n+2)(n+3)$ (4-tetrahedral numbers).
  - *The pattern above continues such that the k^{th} diagonal contains the sequence of order k: $T(n) = \frac{(n+k-1)!}{(n-1)!k!}$ (k-tetrahedral numbers).*
- If you shade every multiple of 2, you will generate a fractal pattern known as the Sierpinski Triangle.
  - *Similar patterns can be generated by shading every multiple of any integer. Also, you can generate this without actually working out the numbers of Pascal's triangle by simply making use of the results $Odd + Odd = Even$, $Odd + Even = Odd$ and $Even + Even = Even$.*
- If the first number (after the 1) in any row is prime, it is a factor of every number in that row > 1.

Most common applications:

1. In probability theory, when calculating combinations, the numbers from Pascal's triangle crop up a lot. If you toss 5 coins, the probability of getting 0, 1, 2, 3, 4 or 5 tails is: $\frac{1}{32}, \frac{5}{32}, \frac{10}{32}, \frac{10}{32}, \frac{5}{32}$ and $\frac{1}{32}$.

In a different scenario, the number of ways of choosing 4 objects from 6 is the 4^{th} number of the 6^{th} row (counting the top row as the 'zeroth' row, and the left-most number in the row as the 'zeroth' number).

2. When multiplying out brackets such as $(x+1)^3$, the coefficients of successive powers of x follow the numbers in the appropriate row of Pascal's Triangle.

Eg:

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

This not only saves a lot of time when calculating results such as $(3x-2)^{20}$, but even extends to negative or fractional powers, allowing us to calculate things like $\sqrt{5}$ from an infinite series (the further we go, the more accurate the approximation):

$$(x+1)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots$$

$$\sqrt{5} = 2\sqrt{\frac{5}{4}} = 2\left(\frac{1}{4}+1\right)^{\frac{1}{2}} \approx 2\left\{1 + \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{8}\left(\frac{1}{4}\right)^2 + \frac{1}{16}\left(\frac{1}{4}\right)^3 - \frac{5}{128}\left(\frac{1}{4}\right)^4 + \frac{7}{256}\left(\frac{1}{4}\right)^5 - \cdots\right\} = 2.236076\ldots$$

Challenges

- How much can you tell me about the numbers of the 100th row of Pascal's Triangle?

The numbers can be found using the general formula $\frac{n!}{r!(n-r)!}$, so it begins with:

$$1 \quad 100 \quad 4950 \quad 161700 \quad \ldots$$

… and ends the same way, since each row is symmetrical.
The numbers will add up to 2^{100} (roughly 1.26×10^{30}).

Multiply out the brackets in the expression $(x+1)^{10}$.

Using numbers from the 10^{th} row:

$$x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1$$

- Four people are to be selected at random from a class of 12 to compete in a challenge. How many different four-person teams are possible?

The fourth number from the twelfth row gives: 495 possible teams.

Extension

Try starting a triangle with the same row-by-row rules, but with 1 2 on the second row instead of 1 1. What patterns remain the same? What changes?

Each row will still have double the total of the last (see original proof). But the linear, quadratic, cubic, etc sequences will have different n^{th} terms.

$$
\begin{array}{c}
1 \\
1 \quad 2 \\
1 \quad 3 \quad 2 \\
1 \quad 4 \quad 5 \quad 2 \\
1 \quad 5 \quad 9 \quad 7 \quad 2 \\
1 \quad 6 \quad 14 \quad 16 \quad 9 \quad 2 \\
1 \quad 7 \quad 20 \quad 30 \quad 25 \quad 11 \quad 2 \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{array}
$$

Notice, for instance, that the diagonal starting 1, 4 is now the square numbers.