Optimisation

One of the most useful applications of differentiation is optimisation.

It allows us to exactly calculate the ideal size for a tin or a box to minimise surface area, and therefore minimise the cost of production.

Dimensions

Baked Beans:
$Radius = 3.8cm$
$Height = 11cm$

Tuna:
$Radius = 4.3cm$
$Height = 3.5cm$

1. Calculate the volume and surface area of the baked beans tin, to 2 significant figures.
Hint: consider the net of a cylinder.

2. Find an expression for the surface area of a cylinder of radius r and volume $500cm^3$.
Hint: start by writing the height in terms of r.

3. Use differentiation to find the optimal radius and the resulting minimum surface area.
Hint: stationary points occur when the derivative equals 0.

4. Calculate the potential saving in steel if tuna were to be sold in optimal cylindrical tins (of the same volume) rather than using current dimensions.
Hint: use scale factors to generalise the result you've already found.

Extension:
The majority of tins are cylindrical in shape – an optimal circular prism (cylinder with equal diameter and height) has a surface area almost 8% smaller than that of an optimal rectangular-based prism (cube). However, even among cylindrical packaging, some are far from optimal (eg Pringles tubes, spice jars, olive oil bottles). Why is this?

Optimisation Solutions

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Dimensions

Baked Beans:
Radius = 3.8*cm*
Height = 11*cm*

Tuna:
Radius = 4.3*cm*
Height = 3.5*cm*

1. Calculate the volume and surface area of the baked beans tin, to 2 significant figures.
Hint: consider the net of a cylinder.

$$V = \pi r^2 h = \mathbf{500cm^3\ to\ 2\ s.f.} \qquad S = 2\pi r^2 + 2\pi r h = \mathbf{350cm^2\ to\ 2\ s.f.}$$

2. Find an expression for the surface area of a cylinder of radius r and volume $500cm^3$.
Hint: start by writing the height in terms of r.

$$V = \pi r^2 h = 500 \quad \Rightarrow \quad h = \frac{500}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right) = \mathbf{2\pi r^2 + \frac{1000}{r}}$$

3. Use differentiation to find the optimal radius and the resulting minimum surface area.
Hint: stationary points occur when the derivative equals 0.

$$S = 2\pi r^2 + \frac{1000}{r} = 2\pi r^2 + 1000r^{-1} \quad \Rightarrow \quad \frac{dS}{dr} = 4\pi r - 1000r^{-2}$$

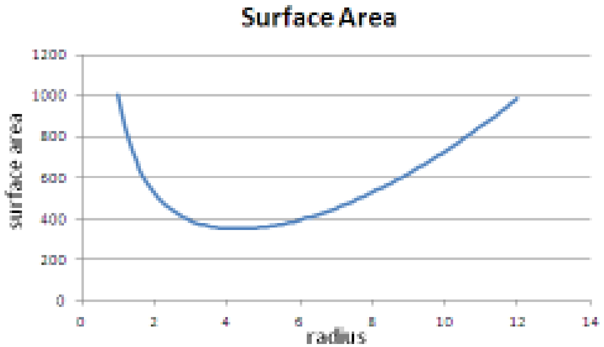
$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} = 0 \quad \Rightarrow \quad 4\pi r^3 = 1000 \quad \Rightarrow \quad r = \frac{10}{\sqrt[3]{4\pi}} = \mathbf{4.30cm\ to\ 3\ s.f.}$$

$$At\ r = \frac{10}{\sqrt[3]{4\pi}} \quad S = 2\pi(4.3\ldots)^2 + \frac{1000}{(4.3\ldots)} = \mathbf{349cm^2\ to\ 3\ s.f.}$$

$$Nature\ of\ stationary\ point: \ \frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3} > 0\ for\ all\ r > 0 \quad \Rightarrow \quad \mathbf{minimum}$$

Note:

This surface area is very close to that of a baked bean tin, although there is a whole centimetre difference in the diameter. In fact, the radius can vary between 3.6 and 5cm without increasing the surface area by more than 3%.



4. Calculate the potential saving in steel if tuna were to be sold in optimal cylindrical tins (of the same volume) rather than using current dimensions.
Hint: use scale factors to generalise the result you've already found.

$$Volume\ of\ regular\ tuna\ tin = 203cm^3\ to\ 3\ s.f.$$

$$Surface\ Area\ of\ regular\ tuna\ tin = 211cm^2\ to\ 3\ s.f.$$

	Optimal cylinder	Optimal tuna tin
Volume	$500cm^3$	$203cm^3$
Surface Area	$349cm^2$	

$$Volume\ scale\ factor = \frac{203}{500} = 0.406 \quad \Rightarrow \quad Linear\ scale\ factor = \sqrt[3]{0.406} = 0.740\ldots$$

$$\Rightarrow \quad Area\ s.f. = (0.740\ldots)^2 = 0.548\ldots \quad \Rightarrow \quad S = 349 \times 0.548\ldots = 191cm^2\ to\ 3\ s.f.$$

$$Saving: \quad \frac{191}{211} = 0.91\ to\ 2\ s.f. \quad \Rightarrow \quad 9\%\ saving$$

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Companies are primarily concerned with maximising profit, and while minimising costs (including packaging costs) is part of that, there is often a trade-off with increased sales and brand recognition due to distinctive shapes and styles. Specific products and how they are used also affects the shape of packages (cereal boxes could use 32% less cardboard by making them spherical, but they would cost more to manufacture, be less efficient to store and transport, and be a real pain to pour from).