AQA

ASSESSMENT and QUALIFICATIONS ALLIANCE

Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Core 4

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for C4

Everything you need to know for Core 4 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

Contents

Chapter 1 – Binomial series expansion

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad for\ |x| < 1$$

Note: This result can be found by considering the geometric series with first term 1 and common ratio x. It is only valid for $-1 < x < 1$ since the condition for S_∞ to exist is that the common ratio $|r| < 1$.

In general, for $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 + \cdots$$

Note: **This formula is in the formula book, but is included here for completeness.**

Note: The alternative form of the binomial series expansion formula (also given in the formula book) is only valid for positive whole values of n.

The binomial series expansion will have an **infinite** number of terms when n **is not a positive integer**.

The binomial series expansion will have a **finite** number of terms when n **is a positive integer**, with the highest power of x being x^n.

The **expansion** of $(1+bx)^n$ is **valid** when $|bx| < 1$. Equivalently, when $|x| < \frac{1}{|b|}$.

Note: This result comes from the original expansion condition, replacing x with bx.

The **expansion** of $(a+x)^n$ is **valid** when $|x| < a$ for any positive constant a.

To **calculate** the expansion of $(a+x)^n$, use $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$

Eg:

$$(4+2x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}} = 2\left(1+\frac{x}{2}\right)^{\frac{1}{2}} = 2\left[1 + \frac{1}{2}\left(\frac{x}{2}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)^2}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^3}{3!} + \cdots\right]$$

$$\approx \mathbf{2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{32}} \ \ \boldsymbol{up\ to\ the\ fourth\ term}$$

Chapter 2 – Rational functions and division of polynomials

To **simplify** rational expressions:

1. Factorise all algebraic expressions.
2. Cancel any factors that are common to the numerator and denominator.

Eg:

$$\frac{x^2+x-6}{3x^2-12} = \frac{(x+3)(x-2)}{3(x+2)(x-2)} = \boldsymbol{\frac{x+3}{3(x+2)}}$$

To **multiply** rational expressions:

1. Factorise all algebraic expressions.
2. Write as a single fraction.
3. Cancel any factors that are common to the numerator and the denominator.

To **divide** rational expressions:

1. Convert the division to a multiplication of the reciprocal.
2. Follow the instructions above.

Eg:

$$\frac{2x+3}{x^2-x} \div \frac{4x^2-9}{(x-1)^2} = \frac{2x+3}{x^2-x} \times \frac{(x-1)^2}{4x^2-9}$$

$$= \frac{(2x+3)(x-1)^2}{(x^2-x)(4x^2-9)} = \frac{(2x+3)(x-1)^2}{x(x-1)(2x+3)(2x-3)} = \boldsymbol{\frac{x-1}{x(2x-3)}}$$

To **add or subtract** rational expressions:

1. Factorise all algebraic expressions.
2. Write each rational expression with the same denominator.
3. Add or subtract to get a single rational expression.
4. Simplify and factorise the numerator.
5. Cancel any factors common to the denominator and the numerator.

Eg:

$$\frac{1}{x+2} - \frac{3}{(x+2)^2} = \frac{(x+2)-3}{(x+2)^2} = \boldsymbol{\frac{x-1}{(x+2)^2}}$$

$$\boldsymbol{P(x) = (ax + b)Q(x) + R}$$

$\boldsymbol{P(x)}$ is a polynomial (degree n).
$(\boldsymbol{ax + b})$ is the divisor (degree 1).
$\boldsymbol{Q(x)}$ is the quotient (degree $n - 1$).
$\boldsymbol{R}$ is the remainder (degree 0).

Eg:

$$4x^3 - 3x^2 + 3x + 3 = (2x + 1)(2x^2 - 2.5x + 2.75) + 0.25$$

When **dividing a polynomial** by a linear expression the remainder will be a constant and the quotient will always be one degree lower than the polynomial.

To apply **algebraic long division**, first write the polynomial and divisor in descending powers of x, including any missing powers of x with a 0 coefficient if necessary (as place-holders).

The **factor theorem** can be extended for factors of the form $(ax + b)$:

$$(ax + b) \text{ is a factor of } P(x) \iff P\left(-\frac{b}{a}\right) = 0$$

More generally, the **remainder theorem** states that:

$$\text{If a polynomial } P(x) \text{ is divided by } (ax + b), \text{the remainder will be } P\left(-\frac{b}{a}\right)$$

Eg:

$$\text{When } 6x^3 - 5x^2 + 2x + 5 \text{ is divided by } (2x + 1) \text{ the remainder is:}$$

$$P\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right)^3 - 5\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 5 = \mathbf{2}$$

Chapter 3 – Partial fractions and applications

Partial fractions is a technique for splitting up fractions, usually to allow a rational function to be integrated or for its binomial expansion to be calculated.

Only **proper** fractions can be written as partial fractions. If a fraction is improper (the degree of the top is not less than the degree of the bottom), it must first be split using division techniques from the previous chapter.

It is necessary to use the correct form of partial fractions in order to correctly calculate terms. The following functions must be split as shown:

$$\frac{p(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\frac{q(x)}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

Note: Usually in exam questions the form of the partial fractions will be provided (and as a result may well involve slightly different functions to the examples shown above).

To find the unknown constants, first add the fractions from the right hand side to produce a fraction with the same denominator as the left. Then either simplify the numerators and compare coefficients, or (usually more efficient) substitute chosen values of x into the identity to eliminate constants.

Eg:

$$\frac{2x}{(x+4)(2x-1)^2} \equiv \frac{A}{x+4} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

$$\frac{2x}{(x+4)(2x-1)^2} \equiv \frac{A(2x-1)^2}{(x+4)(2x-1)^2} + \frac{B(x+4)(2x-1)}{(x+4)(2x-1)^2} + \frac{C(x+4)}{(x+4)(2x-1)^2}$$

$$\Longrightarrow \quad 2x \equiv A(2x-1)^2 + B(x+4)(2x-1) + C(x+4)$$

$$Set\ x = -4: \quad -8 = A(-9)^2 \quad \Longrightarrow \quad A = -\frac{8}{81}$$

$$Set\ x = \frac{1}{2}: \quad 1 = \frac{9}{2}C \quad \Longrightarrow \quad C = \frac{2}{9}$$

$$(no\ values\ of\ x\ exist\ which\ eliminate\ A\ and\ C\ but\ not\ B, so\ substitute\ something\ simple\ next)$$

$$Set\ x = 0: \quad 0 = A - 4B + 4C \quad \Longrightarrow \quad 0 = -\frac{8}{81} - 4B + \frac{8}{9} \quad \Longrightarrow \quad B = \frac{16}{81}$$

$$\Longrightarrow \quad \frac{2x}{(x+4)(2x-1)^2} \equiv \frac{-\frac{8}{81}}{x+4} + \frac{\frac{16}{81}}{2x-1} + \frac{\frac{2}{9}}{(2x-1)^2} \equiv -\boldsymbol{\frac{8}{81(x+4)}} + \boldsymbol{\frac{16}{81(2x-1)}} + \boldsymbol{\frac{2}{9(2x-1)^2}}$$

After splitting a rational function into partial fractions, it is often necessary to **integrate**. The following two results are the most commonly used in this case:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a(ax+b)} + C$$

Eg:

$$\int \frac{4x^2+3x+3}{(3x+2)(x-1)^2}\,dx = \int \frac{1}{3x+2}\,dx + \int \frac{1}{(x-1)}\,dx + \int \frac{2}{(x-1)^2}\,dx$$

(Note: result obtained by partial fractions – working omitted for the purposes of this example)

$$= \mathbf{\frac{1}{3}\ln|3x+2| + \ln|x-1| - \frac{2}{x-1} + C}$$

Another common application of partial fractions is **binomial expansion**. The following two common results are easily derived from the binomial formula (in the formula book), but may be useful to memorise in their own right:

$$(1+y)^{-1} = 1 - y + y^2 - y^3 + y^3 - \cdots \quad valid\ for\ |y| < 1$$

$$(1+y)^{-2} = 1 - 2y + 3y^2 - 4y^3 + \cdots \quad valid\ for\ |y| < 1$$

Eg:

$$The\ binomial\ expansion\ of: \frac{2+3x}{(1+x)(1+2x)}$$

$$\frac{2+3x}{(1+x)(1+2x)} = \frac{1}{1+x} + \frac{1}{1+2x}$$

$$= (1+x)^{-1} + (1+2x)^{-1} = 1 - x + x^2 - \cdots + 1 - (2x) + (2x)^2 - \cdots$$

$$\approx \mathbf{2 - 3x + 5x^3}$$

Chapter 4 – Implicit differentiation and applications

Most functions we have dealt with are explicitly defined (y in terms of x, for instance). Sometimes a function is difficult or impossible to write as $y = f(x)$, and so they are written **implicitly**.

Eg:

$$xy - y^3 = x^2 + 3$$

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$$

$$\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx} \quad where\ f'(y) = \frac{df}{dy}$$

Note: Frequently techniques such as chain rule or product rule are required in conjunction with these rules.

Eg:

$$xy - y^3 = x^2 + 3$$

$$\Longrightarrow \quad x\frac{dy}{dx} + y - 3y^2\frac{dy}{dx} = 2x$$

If necessary, this can be rearranged to give $\frac{dy}{dx}$:

$$\boldsymbol{\frac{dy}{dx} = \frac{2x - y}{x - 3y^2}}$$

Chapter 5 – Parametric equations for curves and differentiation

Often functions can be expressed more simply and usefully in terms of a third variable.

To find the gradient of a curve defined parametrically in terms of t, use:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Problems will usually involve a curve defined parametrically, using or finding specific points on the curve and finding the gradient (or identifying stationary points).

Eg:

$$x = 3\sin t \qquad y = \cos t \qquad \textit{Find the equation of the tangent at the point } t = \frac{\pi}{4}$$

$$\frac{dx}{dt} = 3\cos t \quad \frac{dy}{dt} = -\sin t \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{\sin t}{3\cos t} = -\frac{\sin\frac{\pi}{4}}{3\cos\frac{\pi}{4}} = -\frac{1}{3}\tan\frac{\pi}{4} = -\frac{1}{3}$$

$$x_1 = 3\sin\frac{\pi}{4} = \frac{3\sqrt{2}}{2} \quad y_1 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad m = -\frac{1}{3}$$

$$y - y_1 = m(x - x_1) \quad \Longrightarrow \quad y - \frac{\sqrt{2}}{2} = -\frac{1}{3}\left(x - \frac{3\sqrt{2}}{2}\right) \quad \Longrightarrow \quad \boldsymbol{y = -\frac{x}{3} + \sqrt{2}}$$

To find the Cartesian equation of a curve from its parametric equations, rearrange to eliminate t.

Eg:

$$x = 3\sin t \qquad y = \cos t \quad \Longrightarrow \quad \left(\frac{x}{3}\right)^2 = \sin^2 t \quad y^2 = \cos^2 t$$

$$\Longrightarrow \quad \boldsymbol{\frac{x^2}{9} + y^2 = 1}$$

Chapter 6 – Further trigonometry with integration

Compound angle identities can be used to manipulate trigonometric functions with different input values.

The six identities can be summarised as shown:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \quad \left[A \pm B \neq \left(k + \frac{1}{2}\right)\pi\right]$$

Note: These results **are in the formula book**, so you are not required to learn them. They are included here for the sake of completeness.

You **will** need to know (or be able to derive from the formulae above) the double angle formulae:

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Note: These can readily be derived from the $A \pm B$ formulae.

Eg:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$Let\ A = B \quad \Rightarrow \quad \sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\Rightarrow \quad \sin 2A = 2 \sin A \cos A$$

To integrate either $\sin^2 x$ or $\cos^2 x$ write in terms of $\cos 2x$.

Note: The second and third versions of the $\cos 2A$ formula given above are derived from the first by using the identity $\sin^2 A + \cos^2 A = 1$.

Eg:

$$\int \cos^2 x \, dx = \int \frac{\cos 2x + 1}{2} dx = \boldsymbol{\frac{1}{4}\sin 2x + \frac{x}{2} + C}$$

Functions of the form $a \sin\theta + b\cos\theta$ can be written in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$.

$$R = \sqrt{a^2 + b^2}$$

Note: There are formulae for calculating α directly from a and b, but since a change of sign on either the $\sin\theta$ part, $\cos\theta$ part or both will cause equivalent changes in the formulae, they are not included here. The method described below allows for any variation of form, converting into any of the four varieties of simplified function.

Step 1: Calculate R using $R^2 = a^2 + b^2$.
Step 2: Choose a function to convert into: ($R\sin(\theta + \alpha)$, $R\sin(\theta - \alpha)$, $R\cos(\theta + \alpha)$ or $R\cos(\theta - \alpha)$.
Step 3: Expand this using the appropriate $\sin(A \pm B)$ or $\cos(A \pm B)$ formulae (in formula book).
Step 4: Compare coefficients on the left and right sides of the identity and produce equations for α.
Step 5: Solve for α, using non-primary solutions if necessary to ensure the value fits both equations.

Eg:

Write $3\sin\theta - 2\cos\theta$ in the form $R\sin(\theta - \alpha)$

$$R = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

$$\sin(A - B) = \sin A\cos B - \cos A\sin B \quad \Rightarrow \quad \sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

$$\Rightarrow \quad \sqrt{13}\sin(\theta - \alpha) = \sqrt{13}(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = 3\sin\theta - 2\cos\theta$$

$$\Rightarrow \quad \sqrt{13}\cos\alpha\sin\theta - \sqrt{13}\sin\alpha\cos\theta = 3\sin\theta - 2\cos\theta$$

$$\Rightarrow \quad \cos\alpha = \frac{3}{\sqrt{13}} \quad and \quad \sin\alpha = \frac{-2}{-\sqrt{13}} = \frac{2}{\sqrt{13}}$$

$$\Rightarrow \quad \alpha = 33.7° \; to \; 1 \; d.p.$$

$$\Rightarrow \quad 3\sin\theta - 2\cos\theta = \boldsymbol{\sqrt{13}\sin(\theta - 33.7°)}$$

Note: If the primary solutions to the two equations for α do not match, you will need to look at secondary solutions. Sketch the graph and look further afield. There will always be a value for α that fits both equations, although it may be negative or obtuse.

To solve an equation of the form $a\sin x + b\cos x = c$, first write $a\sin x + b\cos x$ in the form $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$, then rearrange and solve for x.

Chapter 7 – Exponential growth and decay

Many situations can be described by some variation of an exponential growth or decay function. You will need to be familiar with the key rules involving logarithms.

$$a^x = b \quad \Longrightarrow \quad x = \frac{\ln b}{\ln a}$$

Proof:

$$a^x = b$$

$$\ln a^x = \ln b$$

$$x \ln a = \ln b$$

(using the rule $\ln p^q = q \ln p$)

$$\boldsymbol{x = \frac{\ln b}{\ln a}}$$

Note: This is equally valid using any other form of logarithm. Eg, $\frac{\ln b}{\ln a} = \frac{\log_{10} b}{\log_{10} a}$.

The formula $x = a \times b^{kt}$, where a, b and k are positive constants, indicates that x is **growing exponentially**.
The formula $x = a \times b^{-kt}$ indicates that x is **decaying exponentially**.

In general, $x = Ae^{kt} \quad \Longrightarrow \quad \frac{dx}{dt} = kx$

Proof:

$$x = Ae^{kt} \quad \Longrightarrow \quad \boldsymbol{\frac{dx}{dt}} = A(ke^{kt}) = k(Ae^{kt}) = \boldsymbol{kx}$$

Conversely, $\frac{dx}{dt} = kx \quad \Longrightarrow \quad x = Ae^{kt}$

Proof:

$$\frac{dx}{dt} = kx \quad \Longrightarrow \quad \int \frac{1}{x} dx = \int k dt \quad \Longrightarrow \quad \ln x = kt + C$$

$$\Longrightarrow \quad \boldsymbol{x} = e^{kt+C} = e^C e^{kt} = \boldsymbol{Ae^{kt}} \; for \; A = e^C$$

Note: The arbitrary constant A in this formula represents the initial quantity or value (since when $t = 0$, $e^{kt} = 1$). The k determines the rate of increase or decrease.

Chapter 8 – Differential equations

A **differential equation** is an equation which involves at least one derivative of a variable with respect to another variable.

Eg:

$$\frac{dy}{dx} = 2x - 3 \quad or \quad x\frac{dx}{dt} = e^t \sin x \quad or \quad \frac{d^2m}{dt^2} + 3\frac{dm}{dt} + 4m = e^t$$

A **first order differential equation** is one in which the highest order of derivative is the first.

Eg:

$$\frac{dy}{dx} = 2x^4 + 3 \quad or \quad x^2\frac{dx}{dt} = \ln t$$

Note: We will be dealing only with first order differential equations in this module.

Certain common statements involving derivatives need to be interpreted in the form of a differential equation:

The **rate of increase** of x is proportional to x: $\frac{dx}{dt} = kx \quad k > 0$

The **rate of decrease** of x is proportional to x: $\frac{dx}{dt} = -kx \quad k > 0$

Note: These statements can be given in a variety of ways, so you will need to be able to recognise a number of different types.

Eg:
The volume of a snowball decreases at a rate proportional to its volume.

$$\frac{dV}{dt} = -kV$$

The **general solution** of the first order differential equation $\frac{dy}{dx} = g(x)h(y)$ is given by:

$\int \frac{1}{h(y)} dy = \int g(x)dx + C$ where C is an arbitrary constant.

Note: This method is known as the 'Separation of Variables' method, since it involves using division or multiplication to rearrange the differential equation so as to have all x terms on the side of the dx and all the y terms on the side with the dy. To solve then it is necessary to be able to integrate each side (not necessarily straightforward; recall substitution, inspection and integration by parts).

A **particular solution** is obtainable from the general solution by substituting in the values of a specific condition (eg $V = 4$ at $t = 0$), and solving to find the value of the arbitrary constant.

Chapter 9 – Vector equations of lines

A vector is a **quantity** with a **direction.** is represented by bold type such as $\boldsymbol{v}$, with a line underneath, $\underline{v}$ (easier for hand-written work), or, for a vector between two points, $\overrightarrow{AB}$.

The **magnitude** (or size) of the vector $\overrightarrow{AB}$ is written $|\overrightarrow{AB}|$ or occasionally just AB. The magnitude of $\boldsymbol{v}$ is written as $|\boldsymbol{v}|$ or occasionally v.

In two dimensions, the vector $\boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ has component a acting in the x direction and component b acting in the y direction. For 3 dimensions, the third component acts in the z direction.

The **vector between points** A and B is denoted $\overrightarrow{AB}$ and can be calculated from the position vectors of A and B (usually written as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively) as follows:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

Eg:

$$Point\ A: (2,3,-1) \quad Point\ B: (-5,0,2) \quad \Longrightarrow \quad \overrightarrow{OA} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \overrightarrow{OB} = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \mathbf{-7} \\ \mathbf{-3} \\ \mathbf{3} \end{bmatrix}$$

The **magnitude** of the 2D vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and the 3D vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are given by:

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2} \qquad\qquad \left\| \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\| = \sqrt{a^2 + b^2 + c^2}$$

Note: This is simply an application of Pythagoras' theorem, and gives the length of the vector.

Eg:

$$\left\| \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = \mathbf{3}$$

The results above can be combined to calculate **the distance between two points** (although this is easy enough to do in two separate steps): The distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\left\| \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A **unit vector** is a vector with a magnitude of 1. A unit vector can be found with the same direction as any vector simply by dividing that vector by its magnitude.

Note: The unit base vectors are $\boldsymbol{i}, \boldsymbol{j}, and\ \boldsymbol{k}$:

$$\boldsymbol{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \boldsymbol{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \boldsymbol{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Eg:

$$A\ unit\ vector\ in\ the\ direction\ of\ \begin{bmatrix}2\\-2\\1\end{bmatrix}\ is\ given\ by:\ \frac{1}{\sqrt{2^2+(-2)^2+1^2}}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix}$$

The **scalar product** or **dot product** of two vectors is a method for multiplying two vectors together to produce a scalar. The definition is given as:

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$
$$where\ \theta\ is\ the\ angle\ between\ the\ two\ vectors$$

To **calculate the dot product** of two vectors is straight-forward in column form:

$$\begin{bmatrix}a\\b\\c\end{bmatrix}\cdot\begin{bmatrix}d\\e\\f\end{bmatrix} = ad + be + cf$$

Note: This result can be proven from the definition using the properties of the unit base vectors.

The dot product is most commonly used **to find the angle between two vectors**. Since this is the case, it can be more helpful to use the following version of the definition:

$$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \cos\theta$$

Eg:

$$The\ angle\ between\ \begin{bmatrix}-5\\0\\2\end{bmatrix}\ and\ \begin{bmatrix}2\\3\\-1\end{bmatrix}\ is\ given\ by:\ \begin{bmatrix}-5\\0\\2\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-1\end{bmatrix} = \left|\begin{bmatrix}-5\\0\\2\end{bmatrix}\right|\left|\begin{bmatrix}2\\3\\-1\end{bmatrix}\right|\cos\theta$$

$$\Rightarrow\quad -10+0-2 = \sqrt{29}\sqrt{14}\cos\theta \quad\Rightarrow\quad \cos\theta = -\frac{12}{\sqrt{29}\sqrt{14}} \quad\Rightarrow\quad \boldsymbol{\theta = 126.6^\circ\ to\ 1\ d.p.}$$

Provided $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors (zero vectors don't have a specific direction):

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad \Leftrightarrow \quad \boldsymbol{a} \textit{ and } \boldsymbol{b} \textit{ are perpendicular}$$

Note: This is because perpendicular vectors are at 90°, giving $\cos\theta = 0$.

The **vector equation of a line** is given in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ where $\boldsymbol{a}$ is the **position vector** (any point on the line) and $\boldsymbol{b}$ is the **direction vector** (any vector with the same direction as the line).

An example might be:

$$\boldsymbol{r} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$$

This goes through the point (3,4,1) and points in the direction of $\begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$

Note: Since any point on the line will suffice as a position vector and any scalar multiple of the direction vector will also point in the same direction, there is no limit to the number of different equations that could be produced for the same line.

To find the **point of intersection** of two lines (if it exists), generate a **general point** for each, and set them equal to each other. The solutions (if consistent) of the resulting three simultaneous equations for λ or μ will determine the point.

Eg:

$$L_1\text{:} \quad r = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \qquad L_2\text{:} \quad r = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -2 \\ -6 \end{bmatrix}$$

$$L_1 \textit{ General Point}: \quad \begin{bmatrix} 2 \\ 1 + 2\lambda \\ 2\lambda \end{bmatrix} \qquad L_2 \textit{ General Point}: \quad \begin{bmatrix} 1 - 2\mu \\ 2 - 2\mu \\ -1 - 6\mu \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 + 2\lambda \\ 2\lambda \end{bmatrix} = \begin{bmatrix} 1 - 2\mu \\ 2 - 2\mu \\ -1 - 6\mu \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} 2 = 1 - 2\mu \\ 1 + 2\lambda = 2 - 2\mu \\ 2\lambda = -1 - 6\mu \end{matrix} \quad \Rightarrow \quad \mu = -\frac{1}{2} \quad \Rightarrow \quad \lambda = 1$$

$$\textit{Results consistent} \quad \Rightarrow \quad \boldsymbol{\textit{lines intersect}}$$

$$\textit{Substituting } \lambda = 1 \textit{ into the general point for } L_1: \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{2} \end{bmatrix}$$

$$(\textit{Note}: \textit{equivalent to substituting } \mu = -\frac{1}{2} \textit{ into the general point for } L_2)$$

The **angle between two lines** is defined to be the angle between their **direction vectors**.

Eg:

For the lines given above, the angle between would be found by applying the dot product formula to the vectors $\begin{bmatrix}0\\2\\2\end{bmatrix}$ and $\begin{bmatrix}-2\\-2\\-6\end{bmatrix}$.

If two lines are parallel, their direction vectors will be **scalar multiples** of one another. That is, one can be produced by multiplying each element of the other by a particular number.

If two lines are neither **parallel** nor do they **intersect**, they are said to be **skew**.

Note: It is, of course, not possible for a line in two dimensions to be skew. To prove skewness, first show that the lines are not parallel (introduce a scalar variable to multiply by the direction vector and demonstrate that the resulting equations show a contradiction), then show that the lines do not intersect (by constructing simultaneous equations from the general points, as shown previously, and demonstrating that they produce a contradiction).

Eg:

$$Two\ lines\ have\ direction\ vectors\ \begin{bmatrix}6\\4\\4\end{bmatrix}\ and\ \begin{bmatrix}-3\\-2\\2\end{bmatrix}.\ Prove\ that\ they\ are\ not\ parallel.$$

$$If\ parallel:\quad \begin{bmatrix}6\\4\\4\end{bmatrix} = a\begin{bmatrix}-3\\-2\\2\end{bmatrix}\ for\ some\ scalar\ a.$$

$$\Rightarrow \begin{matrix}6 = -3a\\4 = -2a\\4 = 2a\end{matrix} \Rightarrow \begin{matrix}a = -2\\a = -2\\a = 2\end{matrix} \quad \boldsymbol{Not\ consistent} \Rightarrow \boldsymbol{Not\ parallel}$$

If it has already been demonstrated that these two lines do not intersect, these two results are sufficient to conclude the lines must be **skew**.

To find the **shortest distance from a point to a line**, the following steps must be followed:

Step 1: Construct the general point of the line (note: this will involve a λ or μ).
Step 2: Find the vector between your point and this general point (will also involve λ or μ).
Step 3: Find the dot product of this vector with the direction vector of the line, and set equal to 0.
Step 4: Use the value of λ or μ found in step 3 to determine the vector between your point and the general point precisely (just numbers now).
Step 5: Find the magnitude of this vector using Pythagoras' Theorem.

Eg:

Find the shortest distance from the point Q: (3,0,5) to the line L: $r = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

$$The general\ point\ of\ the\ line\ is\ given\ by:\ P = \begin{bmatrix} 1-\mu \\ 2+3\mu \\ -2+2\mu \end{bmatrix}$$

$$The\ vector\ \overrightarrow{PQ}\ is\ given\ by:\ \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 1-\mu \\ 2+3\mu \\ -2+2\mu \end{bmatrix} = \begin{bmatrix} 2+\mu \\ -2-3\mu \\ 7-2\mu \end{bmatrix}$$

$$The\ vector\ \overrightarrow{PQ}\ must\ be\ perpendicular\ to\ the\ line, therefore\ to\ the\ direction\ vector:$$

$$\begin{bmatrix} 2+\mu \\ -2-3\mu \\ 7-2\mu \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = 0 \quad \Rightarrow \quad -2-\mu-6-9\mu+14-4\mu = 0 \quad \Rightarrow \quad \mu = \frac{3}{7}$$

$$From\ this\ value\ of\ \mu, the\ vector\ \overrightarrow{PQ}\ can\ now\ be\ found:\ \overrightarrow{PQ} = \begin{bmatrix} 2+\mu \\ -2-3\mu \\ 7-2\mu \end{bmatrix} = \begin{bmatrix} \frac{17}{7} \\ -\frac{23}{7} \\ \frac{43}{7} \end{bmatrix}$$

$$And\ from\ this, the\ \textbf{distance}\ from\ the\ point\ to\ the\ line, |\overrightarrow{PQ}|, can\ be\ found:$$

$$|\overrightarrow{PQ}| = \left\| \begin{bmatrix} \frac{17}{7} \\ -\frac{23}{7} \\ \frac{43}{7} \end{bmatrix} \right\| = \frac{1}{7}\sqrt{17^2 + (-23)^2 + 43^2} = \frac{\boldsymbol{\sqrt{2667}}}{\boldsymbol{7}} = 7.38\ to\ 3\ s.f.$$

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