Money Matters

Alice and Bob have just accepted jobs at two different firms, and are comparing their choices.

Alice's job pays a starting salary of £20,000 a year, with an automatic annual pay increase of £1500.

Bob's job pays a starting salary of £35,000 a year, with an automatic annual pay increase of £1000.

Both of them expect to remain with the same company until retirement.
Assume for the sake of simplicity that the salary is paid in full at the start of each year.

For example:

Year	**Alice's Salary**	***Alice's Running Total***	**Bob's Salary**	***Bob's Running Total***
1	£20,000	£20,000	£35,000	£35,000
2	£21,500	£41,500	£36,000	£71,000
3	£23,000	£64,500	£37,000	£108,000
…	…	…	…	…

Each person's income can be modelled as an arithmetic series. By considering the first term and the common difference for each person, answer the following questions. Take care to choose whether you are finding U_n or S_n in each case.

1.
a) When will Alice's salary first exceed £40,000?

b) When will her total earnings first exceed £500,000?

2.
a) When will Bob's salary first exceed £40,000?

b)
When will his total earnings first exceed £500,000?

3. When will Alice's salary first exceed Bob's?

4. How long would they both have to work for Alice's total earnings to equal Bob's?

Money Matters SOLUTIONS

First, note that, for Alice: $a = 20000$, $d = 1500$ and for Bob: $a = 35000$, $d = 1000$. We will be making use of the two formulae:

$$n^{th}\ term: \quad U_n = a + (n-1)d$$

$$Sum\ of\ the\ first\ n\ terms: \quad S_n = \frac{n}{2}(2a + (n-1)d)$$

1.

a) When will Alice's salary first exceed £40,000?

$$a = 20000 \quad d = 1500 \qquad U_n = a + (n-1)d$$

$$40000 = 20000 + 1500(n-1) \quad \Longrightarrow \quad \frac{40000 - 20000}{1500} + 1 = 14.\dot{3} \quad \Longrightarrow \quad \boldsymbol{15\ years}$$

b) When will her total earnings first exceed £500,000?

$$S_n = \frac{n}{2}(2a + (n-1)d) \quad \Longrightarrow \quad 500000 = \frac{n}{2}\big(40000 + 1500(n-1)\big)$$

$$\Longrightarrow \quad 1000000 = 40000n + 1500n(n-1) \quad \Longrightarrow \quad 10000 = 400n + 15n^2 - 15n$$

$$\Longrightarrow \quad 15n^2 + 385n - 10000 = 0 \quad \Longrightarrow \quad n = 16 \quad \Longrightarrow \quad \boldsymbol{16\ years}$$

2.

a) When will Bob's salary first exceed £40,000?

$$a = 30000 \quad d = 1000 \qquad U_n = a + (n-1)d$$

$$\Longrightarrow \quad 40000 = 30000 + 1000(n-1) \quad \Longrightarrow \quad \frac{40000 - 30000}{1000} + 1 = \boldsymbol{n = 11\ years}$$

b)

When will his total earnings first exceed £500,000?

$$S_n = \frac{n}{2}(2a + (n-1)d) \quad \Longrightarrow \quad 500000 = \frac{n}{2}\big(60000 + 1000(n-1)\big)$$

$$\Longrightarrow \quad 1000000 = 60000n + 1000n(n-1) \quad \Longrightarrow \quad 1000 = 60n + n^2 - n$$

$$\Longrightarrow \quad n^2 + 59n - 1000 = 0 \quad \Longrightarrow \quad n = 13.746 \ldots \quad \Longrightarrow \quad \boldsymbol{14\ years}$$

3. When will Alice's salary first exceed Bob's?

$$20000 + 1500(n-1) = 30000 + 1000(n-1)$$

$$200 + 15n - 15 = 300 + 10n - 10 \quad \Longrightarrow \quad 5n = 105 \quad \Longrightarrow \quad n = 21 \quad \Longrightarrow \quad \boldsymbol{21\ years}$$

4. How long would they both have to work for Alice's total earnings to equal Bob's?

$$\frac{n}{2}\big(60000 + 1000(n-1)\big) = \frac{n}{2}\big(40000 + 1500(n-1)\big)$$

$$n\big(60000 + 1000(n-1) - 40000 - 1500(n-1)\big) = 0$$

$$600 + 10(n-1) = 400 + 15(n-1)$$

$$200 + 10n - 10 = 15n - 15 \quad \Longrightarrow \quad 205 = 5n \quad \Longrightarrow \quad n = 41 \quad \Longrightarrow \quad \boldsymbol{41\ years}$$