Integration by parts

Integration by parts is a direct reversal of the product rule.
By integrating both sides, we get:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

When to use integration by parts

$$\int x^n \sin mx\ dx$$

(make $u = x^n$)

$$\int x^n e^{mx}\,dx$$

(make $u = x^n$)

$$\int x^n \cos mx\ dx$$

(make $u = x^n$)

$$\int x^n \ln mx\ dx$$

(make $u = \ln mx$)

Although most of the time the simplest part (usually x^n) becomes u and the other part becomes $\frac{dv}{dx}$, if $\ln mx$ is involved, this should be made u, since its integral is not easily found.

When **not** to use integration by parts

Method 1 (rearranging first):

$$\int x^2(3-2x)\ dx = \int 3x^2 - 2x^3\ dx = \boldsymbol{x^3 - \frac{x^4}{2} + C}$$

Method 2 (using parts):

$$\int x^2(3-2x)\ dx \quad \Longrightarrow \quad u = 3-2x \qquad \frac{dv}{dx} = x^2 \qquad \frac{du}{dx} = -2 \qquad v = \frac{x^3}{3}$$

$$\int x^2(3-2x)\ dx = (3-2x)\frac{x^3}{3} - \int -\frac{2x^3}{3}\ dx$$

$$= x^3 - \frac{2x^4}{3} + \int \frac{2x^3}{3}\ dx = x^3 - \frac{2x^4}{3} + \frac{x^4}{6} = \boldsymbol{x^3 - \frac{x^4}{2} + C}$$

In general, if it is possible to simplify, this will be more efficient than using parts.

When to use integration by parts **twice**

For $\int x^n \sin mx\ dx$ or $\int x^n \cos mx\ dx$, if the power of x is more than 1 you will find that the simpler integral generated by integration by parts it still too complicated to evaluate directly.

In this case, use integration by parts again to deal with this integral. An integral involving, for instance, x^2e^x will simplify to one involving xe^x, which itself will simplify to one involving e^x, which can then be evaluated directly.

When integration by parts twice takes you back to square one

For $\int e^x \sin x \ dx$ and similar examples, integration by parts will yield a similar integral (such as $\int e^x \cos x \ dx$), but integration by parts a second time will yield the same type again (one in the form $\int e^x \sin x \ dx$). This may seem to be useless, taking you in circles, but by treating the original integral as the variable you want to solve for and rearranging, you can find its value.

Eg:

$$\int e^x \sin x \ dx$$

$$u = e^x \qquad \frac{dv}{dx} = \sin x \qquad \frac{du}{dx} = e^x \qquad v = -\cos x$$

$$\int e^x \sin x \ dx = -e^x \cos x + \int e^x \cos x \ dx \quad (\mathbf{1})$$

……

Integrating the remaining integral by parts:

$$\int e^x \cos x \ dx$$

$$u = e^x \qquad \frac{dv}{dx} = \cos x \qquad \frac{du}{dx} = e^x \qquad v = \sin x$$

$$\int e^x \cos x \ dx = e^x \sin x - \int e^x \sin x \ dx \quad (\mathbf{2})$$

……

Substituting $(\mathbf{2})$ into $(\mathbf{1})$:

$$\int e^x \sin x \ dx = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \ dx\right)$$

$$\int e^x \sin x \ dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \ dx$$

$$2\int e^x \sin x \ dx = -e^x \cos x + e^x \sin x$$

$$\boldsymbol{\int e^x \sin x \ dx = \frac{e^x}{2}(\sin x - \cos x)}$$

Watch out for: any integrals involving $\ln x$. Even the integral $\int \ln x \ dx$ itself can only be readily evaluated by using parts, and setting $u = \ln x$ and $\frac{dv}{dx} = 1$.

Integration by substitution

Substitution is a change of variable, using a transformation of x to a new variable u in order to change the integration to one more easily evaluated.
The transformation may then be reversed to give a solution in terms of x.

When you must use integration by substitution

For AQA Core 3 exams, any question that requires you to use substitution will provide you with a suitable function to substitute.

Depending on the problem, there are a number of slightly different approaches, but they are all a variation on completing the following:

1. Find a link between du and dx.
(usually done by differentiating your expression for u with respect to x)

2. Replace dx with du.
(usually this includes replacing functions of x or u as well)

3. Replace all functions of x with corresponding functions of u.
(usually by directly comparing the integral with your expression for u)

4. Integrate the function of u with respect to u, then sub x back in.
(usually necessary, unless definite integration and limits are also transformed)

The basic form:

$$\textit{Use the substitution } u = 2x + 3 \textit{ to evaluate: } \int x\sqrt{2x+3}\, dx$$

1.

$$u = 2x + 3 \implies \frac{du}{dx} = 2 \implies \frac{1}{2}\, du = dx$$

2.

$$\int x\sqrt{2x+3}\, dx = \frac{1}{2}\int x\sqrt{2x+3}\, du$$

3.

$$\frac{1}{2}\int x\sqrt{2x+3}\, du = \frac{1}{2}\int \frac{u-3}{2}\sqrt{u}\, du = \frac{1}{4}\int u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\, du$$

Note that $x = \frac{u-3}{2}$ is derived directly from $u = 2x + 3$.

4.

$$\frac{1}{4}\int u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\, du = \frac{u^{\frac{5}{2}}}{10} - \frac{u^{\frac{3}{2}}}{2} + C = \boldsymbol{\frac{(2x+3)^{\frac{5}{2}}}{10} - \frac{(2x+3)^{\frac{3}{2}}}{2} + C}$$

The 'derivative is a factor' form:

$$\textit{Use the substitution } u = x^3 - 1 \textit{ to evaluate: } \int 5x^2(x^3-1)^4\,dx$$

1.

$$u = x^3 - 1 \implies \frac{du}{dx} = 3x^2 \implies \frac{1}{3}\,du = x^2\,dx$$

2.

$$\int 5x^2(x^3-1)^4\,dx = \frac{1}{3}\int 5(x^3-1)^4\,du$$

3.

$$\frac{1}{3}\int 5(x^3-1)^4\,du = \frac{1}{3}\int 5(u)^4\,du = \frac{5}{3}\int u^4\,du$$

4.

$$\frac{5}{3}\int u^4\,du = \frac{5}{3}\left[\frac{u^5}{5}\right] + C = \frac{u^5}{3} + C = \boldsymbol{\frac{(x^3-1)^5}{3} + C}$$

The 'x as a function of u' form:

$$\textit{Use the substitution } x = \cos u \textit{ to evaluate: } \int \frac{1}{\sqrt{1-x^2}}\,dx$$

1.

$$x = \cos u \implies \frac{dx}{du} = -\sin u \implies dx = -\sin u\,\,du$$

2.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\int \frac{\sin u}{\sqrt{1-x^2}}\,du$$

3.

$$-\int \frac{\sin u}{\sqrt{1-x^2}}\,du = -\int \frac{\sin u}{\sqrt{1-(\cos u)^2}}\,du = -\int \frac{\sin u}{\sin u}\,du = -\int 1\,du$$

Note that $\sqrt{1-\cos^2 u} = \sin u$ since $\sin^2\theta + \cos^2\theta = 1$.

4.

$$-\int 1\,du = -u + C = -\boldsymbol{\cos^{-1} x + C}$$

Note that, as in this case, it may be not till the end that u in terms of x is needed.

Watch out for: changing limits in a definite integration problem. It is not usually necessary to convert your integral back into a function of x at the end, but if you don't, make sure you change the limits of the integral to limits for u:

Eg: If using $u = \tan x$, $\int_0^2 f(x)\,dx$ becomes $\int_{\tan 0}^{\tan 2} g(u)\,du$

Integration by inspection

Inspection relies on knowledge of the type of function that could be differentiated to get the one we want to integrate. It can be thought of as a special case of substitution, where the substitution is generally for a linear function of x (that is, $u = ax + b$).

When you can use integration by inspection

The most common types are functions involving a linear function of x. You need to be able to recognise the type of function and use the following results:

$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C;\ n \neq -1$	$\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + C$

Eg:

$$\int \sqrt{5-3x} \, dx = -\frac{2}{9}(5-3x)^{\frac{3}{2}} + C \qquad \int e^{2x+1} \, dx = \frac{e^{2x+1}}{2} + C$$

The implied substitution here is $u = ax + b$, and they both yield 'nice' results because $\frac{du}{dx} = a \implies \frac{1}{a}\, du = dx$. This is essentially the reverse of chain rule.

Another example of a commonly required inspection result is:

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

Eg:

$$\int \frac{2x-1}{3x^2-3x+5} \, dx = \frac{1}{3}\int \frac{6x-3}{3x^2-3x+5} \, dx = \frac{1}{3}\ln|3x^2-3x+5| + C$$

The implied substitution here is $u = f(x)$, giving: $\frac{du}{dx} = f'(x) \implies du = f'(x)\, dx$:

$$\int \frac{f'(x)}{f(x)} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|f(x)| + C$$

Watch out for: these two similar-looking but different applications of inspection:

$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\int \frac{a}{ax+b} \, dx$ $= \frac{1}{a}\ln\|ax+b\| + C$	$\int \frac{1}{(ax+b)^2} \, dx = \int (ax+b)^{-2} \, dx$ $= \frac{(ax+b)^{-1}}{-a} + C = \frac{1}{a(ax+b)} + C$

Integration using partial fractions

Partial fractions is a method for writing a complicated fraction as the sum of simpler ones. For instance, $\frac{2}{x(x+2)}$ can be written equivalently as $\frac{1}{x} - \frac{1}{x+2}$.

When you should use partial fractions

For AQA Core 4 exams, particularly for the more tricky cases, the format for partial fractions will be given when you are required to use them.

However, for simple cases you should be confident applying the following:

$\frac{px+q}{(ax+b)(cx+d)}$ can be written in the form $\frac{A}{ax+b} + \frac{B}{cx+d}$

Note that this assumes the denominators are not scalar multiples of one another.

Once a fraction has been split into simpler ones using partial fractions, you can use the inspection results $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C;\ n \neq 1$ and $\int \frac{f'(x)}{f(x)} \, dx$.

When you need to spot partial fractions:

Usually when you have a quadratic denominator that factorises, but the numerator cannot be scaled up or down to give the differential of the denominator (if it can, use $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$ instead). Split the fraction into two, where the denominators are the factors of your original denominator.

Eg:

$$\frac{3x+1}{(2x-1)(x+1)} \equiv \frac{A}{2x-1} + \frac{B}{x+1} \implies \dots \equiv \mathbf{\frac{5}{3(2x-1)} + \frac{2}{3(x+1)}}$$

Note that the details of a partial fractions method are not given here, since we are focusing on how it is applied to integration problems. For more examples of splitting a fraction in this way, see your notes or text book.

Now the integral becomes:

$$\int \frac{5}{3(2x-1)} + \frac{2}{3(x+1)} \, dx = \frac{5}{6}\int \frac{2}{2x-1} \, dx + \frac{2}{3}\int \frac{1}{x+1} \, dx = \boldsymbol{\frac{5}{6}\ln(2x-1) + \frac{2}{3}\ln(x+1) + C}$$

When you are given the partial fractions form to use:

When the denominator has repeated factors, or the fraction is improper, the format used for partial fractions is more complicated, but is usually provided.

Improper fraction (numerator has order greater than or equal to denominator):

Write $\frac{2x^2-x+11}{(2x-3)(x+2)}$ in the form $A+\frac{B}{2x-3}+\frac{C}{x+2}$ and hence find $\int \frac{2x^2-x+11}{(2x-3)(x+2)}\,dx$.

$$2x^2-x+11 \equiv A(2x-3)(x+2)+B(x+2)+C(2x-3)$$

$x=-2 \Rightarrow C=-3$

$x=\frac{3}{2} \Rightarrow B=4$

$x=0 \Rightarrow A=1$

$$\frac{2x^2-x+11}{(2x-3)(x+2)} = \mathbf{1}+\frac{\mathbf{4}}{(\mathbf{2x-3})}-\frac{\mathbf{3}}{\mathbf{x+2}}$$

$$\int \frac{2x^2-x+11}{(2x-3)(x+2)}\,dx = \int 1+\frac{4}{(2x-3)}-\frac{3}{x+2}\,dx$$

$$= \boldsymbol{x+2\ln(2x-3)-3\ln(x+2)+C}$$

Repeated factors in the denominator (eg squared bracket):

Write $\frac{x}{(x+1)(x-1)^2}$ in the form $\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ and hence find $\int \frac{x}{(x+1)(x-1)^2}\,dx$.

$$x \equiv A(x-1)^2+B(x+1)(x-1)+C(x+1)$$

$x=1 \Rightarrow C=\frac{1}{2}$

$x=-1 \Rightarrow A=-\frac{1}{4}$

$x=0 \Rightarrow B=\frac{1}{4}$

$$\frac{x}{(x+1)(x-1)^2} = -\frac{\mathbf{1}}{\mathbf{4(x+1)}}+\frac{\mathbf{1}}{\mathbf{4(x-1)}}+\frac{\mathbf{1}}{\mathbf{2(x-1)^2}}$$

$$\int \frac{x}{(x+1)(x-1)^2}\,dx = \frac{1}{4}\int -\frac{1}{(x+1)}+\frac{1}{(x-1)}+\frac{2}{(x-1)^2}\,dx$$

$$= \boldsymbol{\frac{1}{4}\left(-\ln(x+1)+\ln(x-1)-\frac{2}{x-1}\right)+C}$$

Watch out for: opportunities to use $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$ since these may look like candidates for partial fractions but a quick check will often confirm that the numerator is the differential of the denominator (or can be made into such by multiplying or dividing by a number).

Eg $\int \frac{2x+5}{2x^2+10x+1}\,dx = \frac{1}{2}\int \frac{4x+10}{2x^2+10x+1}\,dx = \boldsymbol{\frac{1}{2}\ln(2x^2+10x+1)+C}$

Integration using standard results

While you may be asked to prove some of these results using a suitable method, unless explicitly asked to do so you may assume all of these throughout your work. Often a more involved integration problem will incorporate one or more of the standard results. They are on pages 7 and 8 of the AQA Formula Book.

What to look for with standard results

While you don't need to memorize these results, you do need to be able to spot them when they show up in questions. Make use of both lists (shown below). Note that the hyperbolic trig results ($sinh$, $cosh$, etc) are for Further Maths only.

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$	$\frac{\mathrm{f}'(x)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}'(x)}{(\mathrm{g}(x))^2}$

Integration

(+ constant; $a > 0$ where relevant)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan(\frac{1}{2}x)\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert = \ln\lvert\tan(\frac{1}{2}x+\frac{1}{4}\pi)\rvert$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$ or $\ln\left\{x+\sqrt{x^2-a^2}\right\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$ or $\ln\left\{x+\sqrt{x^2+a^2}\right\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Watch out for: trig functions (everything except sin and cos are provided, including inverse trig and sec, $cosec$ and cot. Also note the variations involving x^2 in the denominator of a fraction – some rearrangement may be necessary.

Eg:

$$\int \frac{4}{\sqrt{5-x^2}}\,dx = 4\int \frac{1}{\sqrt{\left(\sqrt{5}\right)^2 - x^2}}\,dx = \mathbf{4\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C}$$