Geometric Series

Introduction

A Geometric Series is an infinite sum whose terms follow a geometric progression – that is, each pair of terms is in the same ratio.

A simple example wound be:

$$1 + 2 + 4 + 8 + 16 + . . .$$

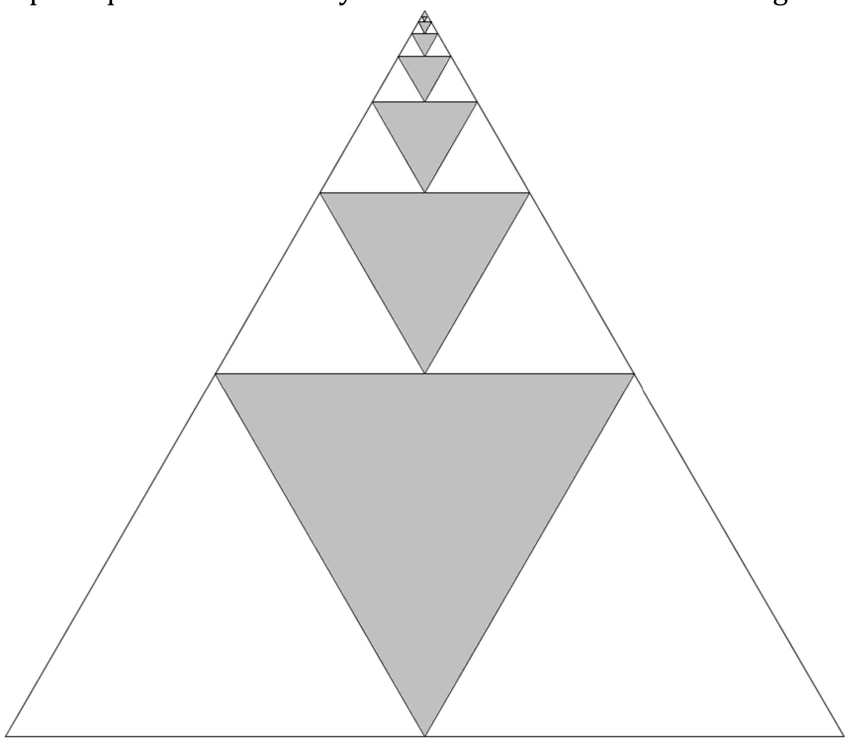
where the first term is 1, and the common ratio is 2. Each term is twice the size of the previous term.

Given the first term and common ratio, a geometric series can be produced, and depending on the common ratio, it may converge to a limit. In fact, for a common ratio strictly between 1 and -1, it will converge. Moreover, we can determine its limit. For instance:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + . . . = 2$$

Geometric Diagrams

A geometric series can perhaps be more readily understood in the form of a diagram:



If the large triangle has an area of 1, the total area of the shaded region can be represented by the geometric series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + . . .$$

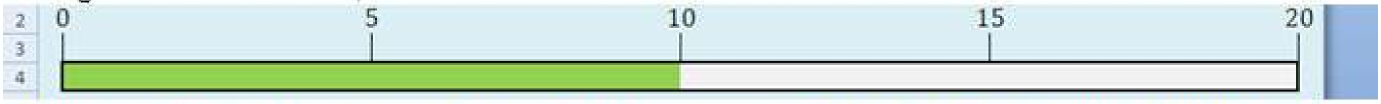
But it is also clear by examining the triangle from the bottom up in layers that the shaded region, in total, comprises exactly one third.

Buffering

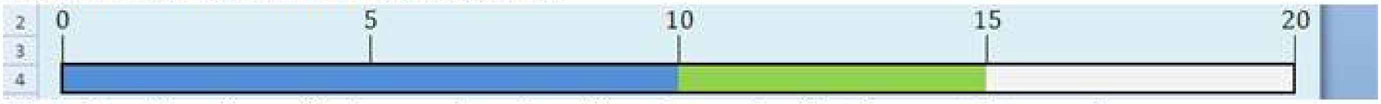
The concept of an infinite series of numbers, none of them zero, adding together to give a finite result, may seem strange, but consider as an example a video that buffers at half the speed it will play:

(Screen shots from [Buffer Simulation](#) spreadsheet)

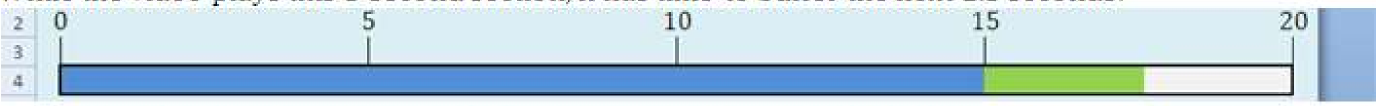
During the first 20 seconds, it buffers 10 seconds' worth of video:



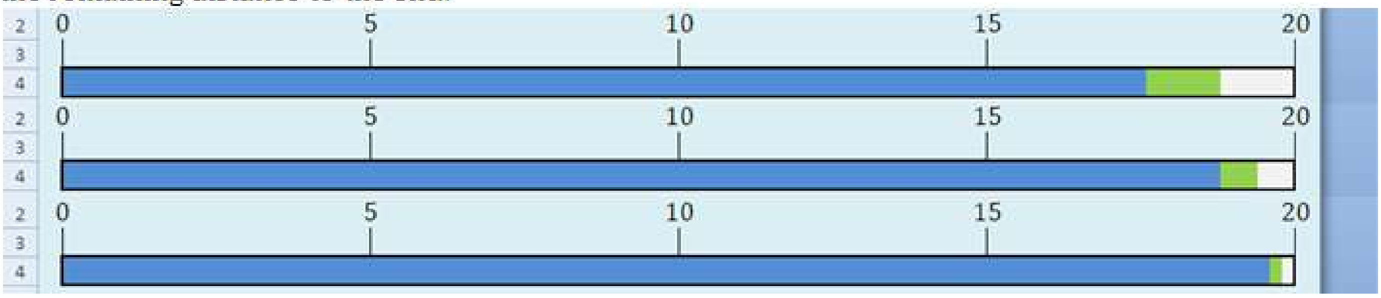
At this point, the video begins to play. In the 10 seconds it takes to play the previously buffered section, the video has buffered the next 5 seconds:



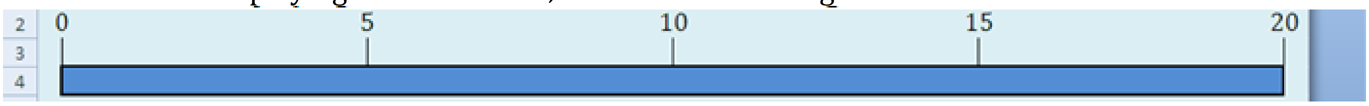
While the video plays this 5 second section, it has time to buffer the next 2.5 seconds:



Every time the video catches up with the previous end-point of the buffer, the buffer has covered half of the remaining distance to the end:



While we could continue this progression forever, in increasingly small steps, the steps take an increasingly small length of time to happen – rapidly approaching no time at all, in fact. And by the time the video has been playing for 20 seconds, it has been buffering for 40 seconds.



If we were to consider this situation from a Geometric Series point of view, we might think about the infinite sum of the lengths of time the video buffered in each step – 20 seconds to buffer the first 10 seconds of video, 10 for the next 5, and so on:

$$20 + 10 + 5 + 2.5 + 1.25 + 0.625 + 0.3125 + \ldots$$

But we already know the answer to this, by using a much simpler approach; the video buffers at half the speed of playback, so the time taken to buffer the full 20 second video has to be 40 seconds.

This is the same result, only scaled up by a factor of 20, as quoted earlier:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 2$$

Formulae

Once the concept of an infinite sum makes some sense, we can start to examine how to calculate this algebraically.

First, however, we must define our terms:

Let a be the first term of the series.
Let r be the common ratio.
Let n be the number of terms (The finite sum of the first n terms will be useful)
Let U_n be the n^{th} term of the series.
Let S_n be the sum of the first n terms.
Let S_∞ be the infinite sum of the series.

Now, since each term is generated from each previous term by multiplying by the common ratio, the general form of a geometric series, up to n terms, will be:

$$a + ar + ar^2 + ar^3 + \ . \ . \ . \ + ar^{n-1}$$

And therefore the n^{th} term will be given by:

$$\boldsymbol{U_n = ar^{n-1}}$$

Now, a simple formula for the sum of the first n terms can be found using a neat trick:

$$S_n = a + ar + ar^2 + ar^3 + \ . \ . \ . \ + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ . \ . \ . \ + ar^n$$

$$S_n - rS_n = a - ar + ar - ar^2 + ar^2 - \ . \ . \ . - ar^{n-1} + ar^{n-1} - ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$$

The sum to infinity is simply derived from this, for the case where $-1 < r < 1$:

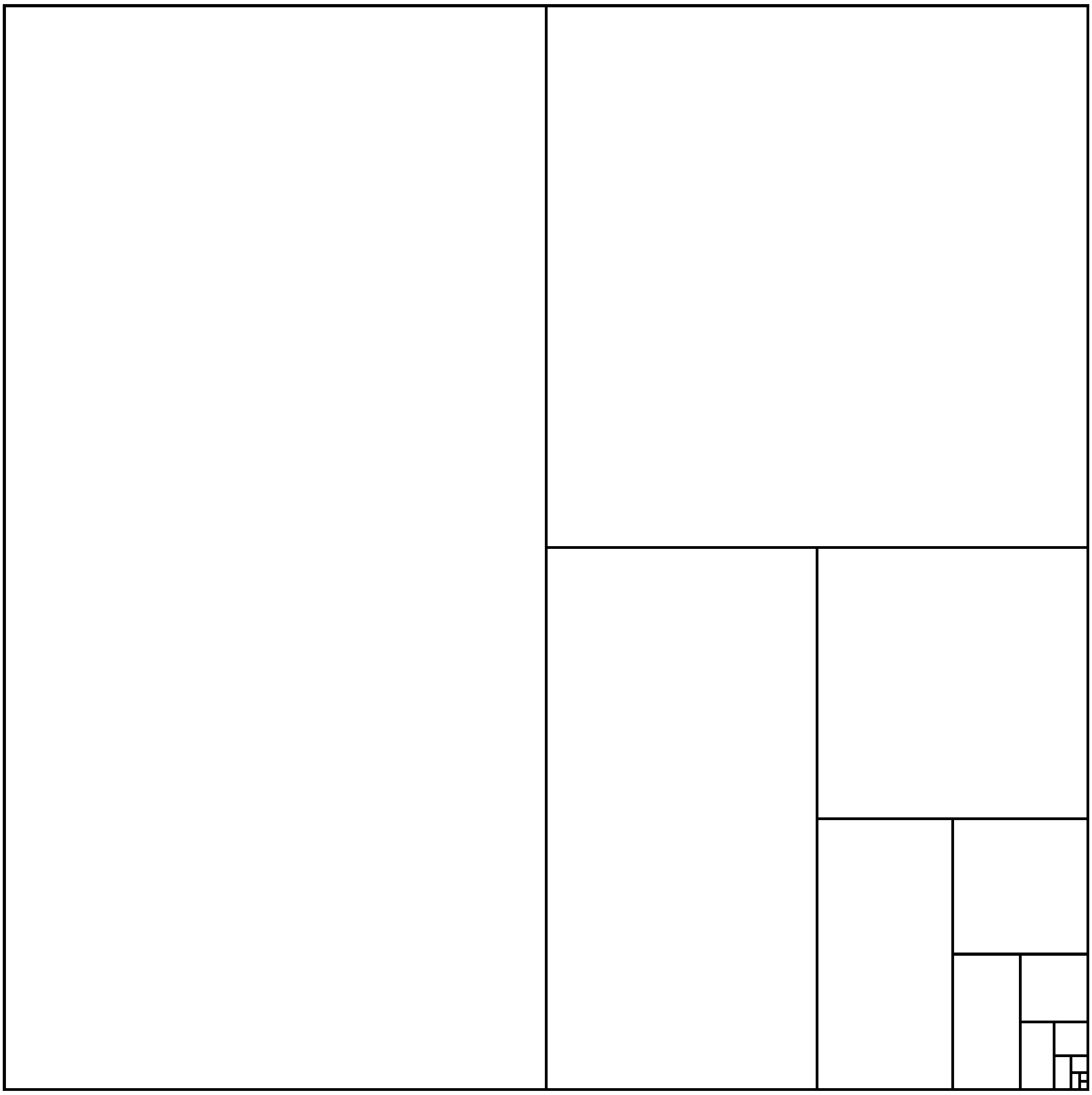
$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$Since - 1 < r < 1, r^n \to 0 \ as \ n \to \infty$$

$$\Rightarrow S_\infty = \frac{a(1-0)}{1-r}$$

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

Geometric Series Problem 1



The large outer square has side length 1. Each line exactly halves the square or rectangle it goes through. This diagram represents an infinite series of divisions.

1. Continue this geometric series for the area of the rectangles to find the next 4 terms:

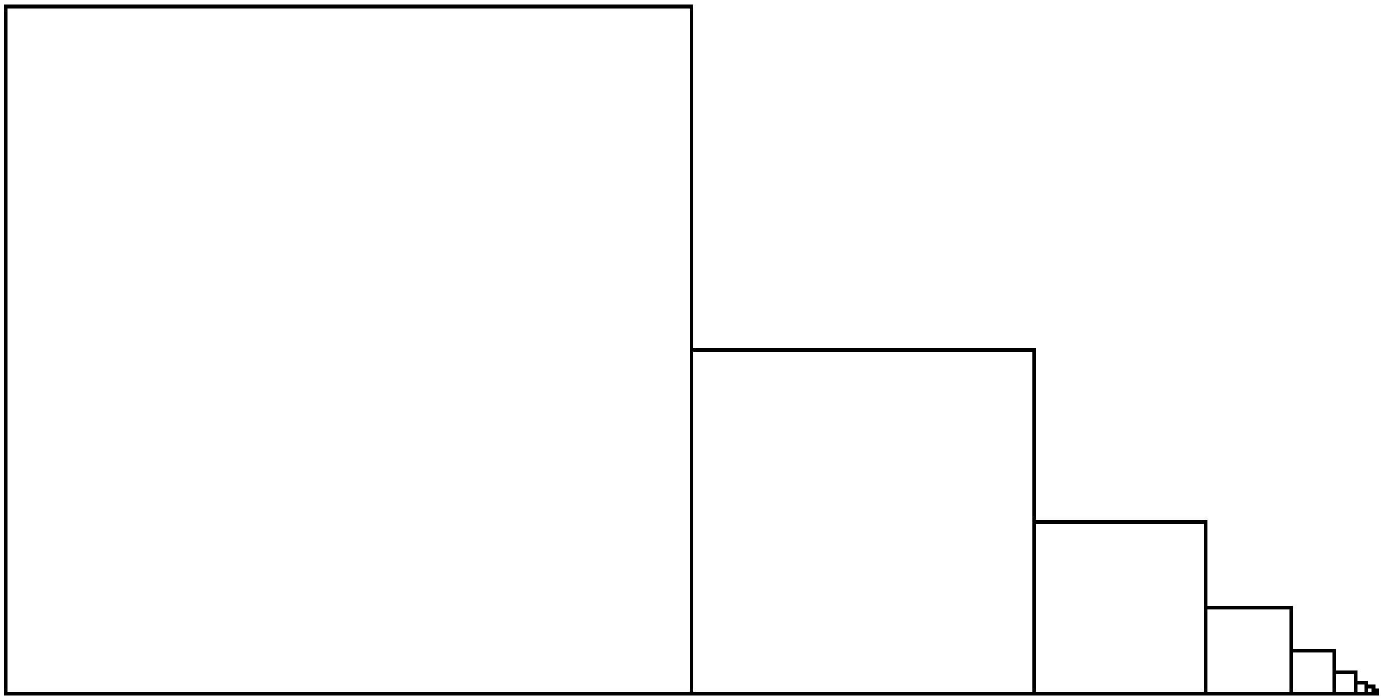
$$\frac{1}{2} + \frac{1}{8} +$$

2. Calculate the area of the first 10 rectangles.

3. Calculate the area of all the rectangles.

4. Find the area of all the squares using geometric series methods, and verify that the total area of the rectangles and squares is 1.

Geometric Series Problem 2

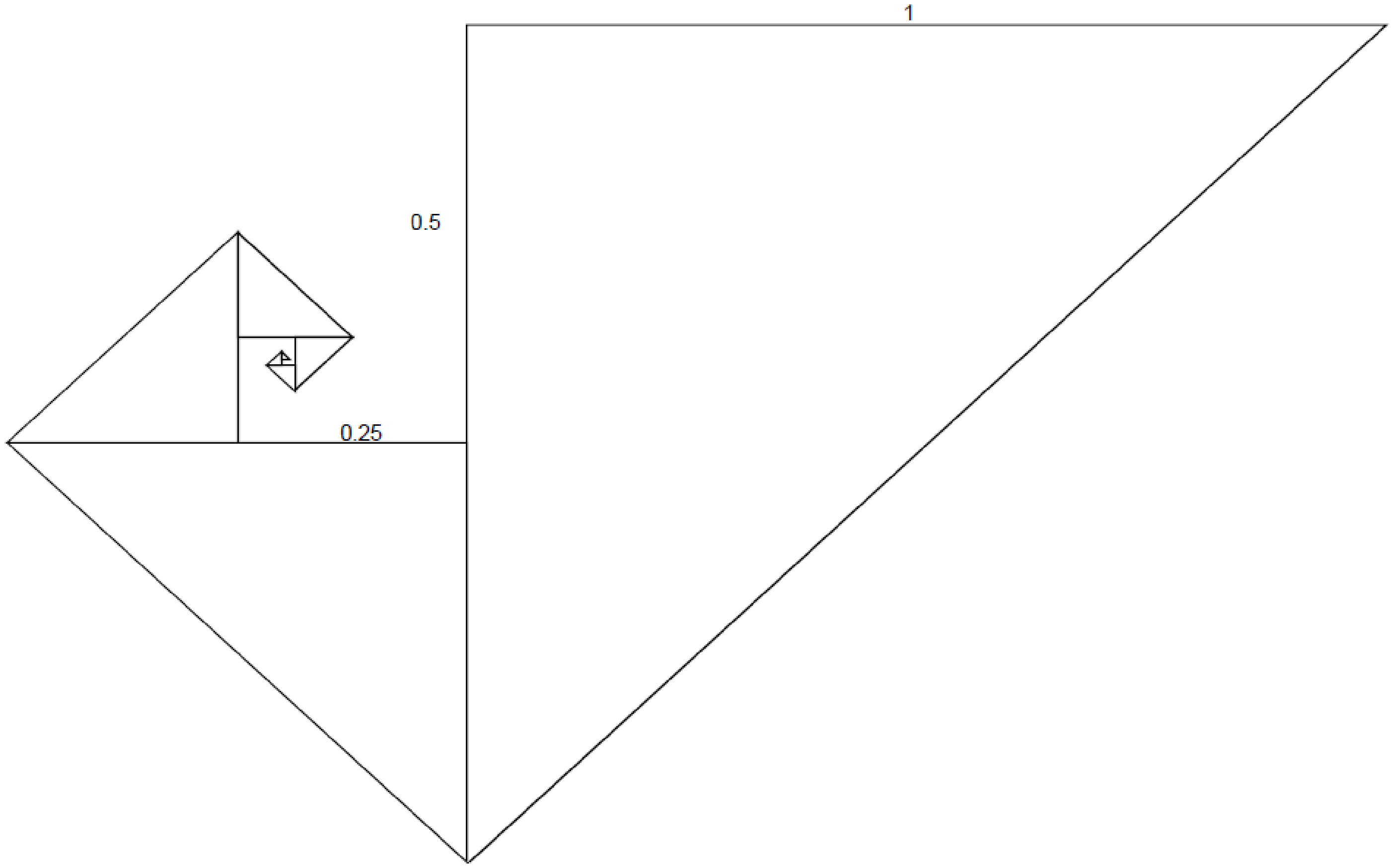


The largest square has side length $\frac{1}{2}$, and each subsequent square is half of the length. The diagram represents an infinite number of squares.

1. Using geometric series methods, calculate the total area of these squares.

2. By considering separately the vertical and horizontal lines, calculate the total perimeter of the shape
   Note: internal lines are not to be included.

Geometric Series Problem 3



1. Using geometric series methods, calculate the total area of this infinite shape.

2. Find the total outer perimeter.

3. How many triangles would need to be included before their combined area was within 1% of the total area of the shape?

Geometric Series Problem 4

A ball is dropped to the ground from a height of 1 metre. It bounces back to 80% of its original height before falling again.

1. By using geometric series methods, calculate the distance the ball has travelled by the time it hits the ground for the 10^{th} time.
Hint: Deal with upward and downward motion separately, then combine the results.

2. Find the total distance the ball will travel if left to bounce indefinitely.

3. Using SUVAT equations, find an expression for the time taken for a ball to reach the ground when dropped, from rest, from a height x metres above the ground.

4. Use your expression to construct an infinite sum, and thus find the total time the ball is in motion.
Hint: Writing down the first few terms of the series should help you to identify the common ratio. You should still treat upward and downward motion separately.

5. Finally, calculate the average speed of the ball while in motion.
Hint: You have the total time and the total distance.