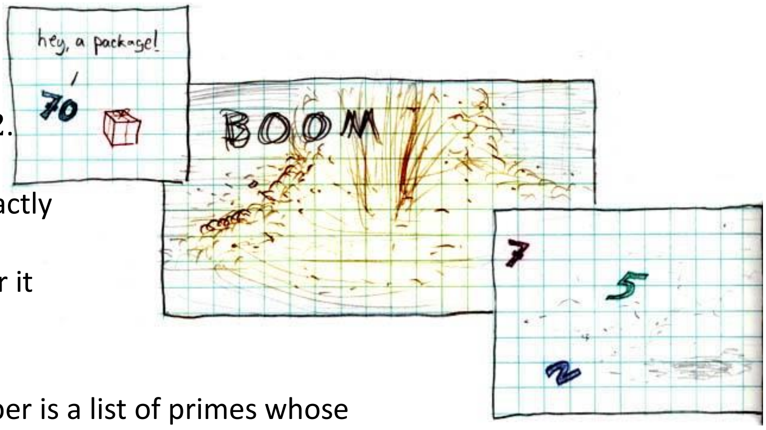
Factors Investigation

The basics

Definition: A **factor** is a positive integer which divides exactly into another number.
Example: 8 is a factor of 72 because $8 \times 9 = 72$.

Definition: A **prime** is a positive integer with exactly two factors (itself and 1).
Example: 349 is a prime because the only factor it has other than 1 (the universal factor) is 349.



xkcd.com

Definition: The **prime decomposition** of a number is a list of primes whose product is equal to the number. Every positive integer greater than 1 can be written uniquely as a product of primes.
Example: $280 = 2^3 \times 5 \times 7$.

The investigation

The aim is to investigate how the *number of factors* is related to the *prime decomposition* of a number.

Getting started

As with any open investigation, conclusions are most readily found through a clear, structured approach which relies on the use of conjectures to direct your efforts. A conjecture is simply an observation which you think may be true. You do not need to be convinced by it, but by clarifying the pattern or rule you think you have observed you will be in a better position to investigate further and hopefully prove or disprove (either is beneficial) your conjecture. If you prove a conjecture correct, see if you can strengthen or extend it. If you prove it false, consider how it might be modified or improved to make it true. If you find a conjecture is partially true, but breaks down in certain cases, consider what makes those cases different and try to develop or modify your conjecture accordingly.

Problem solving

Begin by following these three simple steps:

1. Clarify the question	Even if the problem you are working on is given in deliberately vague terms (such as this one!) you should do your best to narrow down what it is asking of you. • How will you know when you've solved the problem? • What would a satisfactory answer look like? • Will you need to demonstrate how you got your answer?
2. Consider your position	Having a clear view of your starting point will help you see where to go next. • What do you already know that may be helpful? • What relevant information have you been given?
3. Decide on an approach	• When you get stuck, try a different tack. • Use conjectures to test your ideas, and when you think you have found a valid conclusion, try to explain why it must be true. • If the problem is too hard to tackle, try to simplify it. Solve an easier version, then see if you can extend the ideas to the harder one.

(turn over for hints and suggestions, and for extension challenge questions)

Factors Investigation
Hints, suggestions and challenges

Hints & suggestions for when you get stuck

- Generate some numbers with simple prime factorisations first (eg only powers of a single prime).
- Start with small numbers. List all their factors, and write their prime decomposition. Compare those with the same number of factors and look for patterns in the prime decomposition.
- Ask questions such as "What prime factorisations yield numbers with an odd number of factors?"
- Consider closely related numbers. Try modifying a number by multiplying by a single prime, and see what effect that has on the number of factors. Does it make a difference which prime you use?
- Use prime decomposition to generate factors instead of the traditional factor pairs method. How many ways can you produce a factor of a given number, using only its prime factors?

The final challenges

Once you've developed your ideas sufficiently and reached some useful conclusions, you should be able to attempt a few of these (hard) challenges:

(answers may be left in prime decomposition form)

- Find the smallest number with exactly 7 factors.
- Find the smallest number with exactly 17 factors.
- Find all the numbers composed exclusively of single-digit primes with exactly 10 factors.
- Find the smallest number with exactly 100 factors.
- Describe the type of numbers which have a *prime* number of factors.
- How many factors does $10!$ have? Remember $10! = 10 \times 9 \times 8 \times \ldots \times 3 \times 2 \times 1$.
- How many factors does $7^6 + 7^7 + 7^8 + 7^9$ have?
- How many factors does $4^7 \times 6^8 \times 12^6$ have?

What's next?

As you delve into the weird and wonderful world of number patterns – especially as you wrestle with primes and their mysterious ways – you will begin to scratch the surface of such topics as modular arithmetic (which, apart from explaining why the sum of the digits of a multiple of 3 is a multiple of 3, enables mathematicians – even without a calculator – to figure out the last digit of $2^{57885161} - 1$ and is the key component of RSA encryption – the virtual lock that is easy to close but nigh impossible to open. Keep an eye out for a harmless-sounding topic called Number Theory …

Factors Investigation SOLUTIONS

Note: the following is designed to resemble a logical approach to the problem. It is not the *only* logical approach by any means, and – since it does not include dead ends or unproductive conjectures it is both unrealistically short compared to a genuine investigation and unrealistically linear in its progress.

Trial 1: Investigating numbers which are powers of a single prime. Starting by listing the factors of the first few powers of 2.

$Number$:	2	4	8	16	32
$Factors$:	1×2	1×4 2×2	1×8 2×4	1×16 2×8 4×4	1×32 2×16 4×8
$Number\ of\ factors$:	2	3	4	5	6

Conjecture 1: Square numbers, and only square numbers, have an odd number of factors.

Testing: Try square numbers with only one unique prime factor, then some with more.

Proof: Consider the factors of any number. They can be arranged in pairs as shown above. Therefore, for every number which is a factor, there is another number which is uniquely part of its factor pair. This suggests that there will always be an even number of factors. However, this is not the case when a factor pair contains a repeated factor. In this case, our number can be written as $k \times k$, for some whole number k, so it is, by definition, a square number. Conversely, a square number will always have some factor pair with a repeated factor (but never more than one) so it will always have an odd number of factors. Conjecture 1 proven.

Trial 2: There appears to be a logical pattern to the number of factors as we increase the power of 2, so do a similar thing with other primes to see if it represents a more general rule:

Number	Prime Decomposition	Number of factors
2	2	2
4	2^2	3
8	2^3	4
16	2^4	5
32	2^5	6
...	...	...
3	3	2
9	3^2	3
27	3^3	4
81	3^4	5
243	3^5	6
...	...	...
5	5	2
25	5^2	3
125	5^3	4
625	5^4	5
3125	5^5	6

Conjecture 2: The number of factors of a number with only one unique prime factor is always one greater than the power of the prime factor.

Proof: The supporting evidence of a few trials is not sufficient, so we need to consider *why* this seems to be true. In order to do this, we will consider a list of factors, *written in prime factorised form*. We will also use p to represent a general prime number, since this argument should work regardless of the prime chosen. Finally, generalising further, we shall attempt to show why the conjecture must be true.

$32 = 2^5$ 1×2^5 2×2^4 $2^2 \times 2^3$ $2^3 \times 2^2$ $2^4 \times 2$ $2^5 \times 1$ (6 factors)	p^5 $1 \times p^5$ $p \times p^4$ $p^2 \times p^3$ $p^3 \times p^2$ $p^4 \times p$ $p^5 \times 1$ (6 factors)	p^n $1 \times p^n$ $p \times p^{n-1}$ $p^2 \times p^{n-2}$ ... $p^{n-1} \times p$ $p^n \times 1$ ($n + 1$ factors)

Note: writing out all factors twice is the easiest way to extend readily to the more algebraic versions, as well as to more readily see why the rule works.

Trial 3: Begin with a small number and multiply by a different prime factor to see how this affects the total number of factors.

$4 = 2^2 \implies 3\, factors\ (1, 2, 4)$ $12 = 2^2 \times 3 \implies 6\, factors\ (1, 2, 3, 4, 6, 12)$	$7 = 7 \implies 2\, factors\ (1, 7)$ $35 = 5 \times 7 \implies 4\, factors\ (1, 5, 7, 35)$
$15 = 3 \times 5 \implies 4\, factors\ (1, 3, 5, 15)$ $30 = 2 \times 3 \times 5 \implies 8\, factors\ (1, 2, 3, 5, 6, 10, 15, 30)$	$6 = 2 \times 3 \implies 4\, factors\ (1, 2, 3, 6)$ $18 = 2 \times 3^2 \implies 6\, factors\ (1, 2, 3, 6, 9, 18)$

Initially it appeared that including one more prime factor doubled the number of factors. However, the counter example of 6 to 18 means there must be more going on. On inspection, it is clear that for the first three examples the prime being included was different to any previously in the number. For 6 to 18, the prime was not different. These cases will require more consideration, but a limited conclusion seems evident:

Conjecture 3: Multiplying a number by a new prime (one not already present in the prime decomposition) will double the number of factors.

Partial proof: Consider a list of factors for a number pqr where p, q and r represent primes. They would each be a combination of these prime factors. Then consider the new factors made possible by introducing a new, distinct, prime, s:

Original factors:	Newly possible factors:
1	s
p	ps
q	qs
r	rs
pq	pqs
pr	prs
qr	qrs
pqr	$pqrs$

Note that the new number, $pqrs$, will still maintain its original factors, but will also have a new set of factors, one for each of the original factors. Therefore the number of factors is doubled from 8 to 16.

Note that the previous proof depends on all of these factors being unique. Since p, q, r and s were all different prime numbers, this is the case for the example used. It would not necessarily be the case when the new prime to be included is already present in the original decomposition of the number.

Extended proof: The arguments above can readily be extended for any number of primes present in the original number, and also for powers of these primes. Since any factors of the original number can immediately form an entirely new factor simply through multiplication with the new prime, the number of factors is automatically doubled.

Trial 4: The task of investigating changes for a variety of numbers seems somewhat daunting, so I'm going to consider a different way of listing factors, starting with the idea of combinations of primes (look at the table of p, q and r). Consider, for instance, the number $144 = 2^4 \times 3^2$. The options can be summarised in a table:

Factors of 144	$\mathbf{2^0}$	$\mathbf{2^1}$	$\mathbf{2^2}$	$\mathbf{2^3}$	$\mathbf{2^4}$
$\mathbf{3^0}$	1	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$
$\mathbf{3^1}$	$3^1 = 3$	$2^1 \times 3^1 = 6$	$2^2 \times 3^1 = 12$	$2^3 \times 3^1 = 24$	$2^4 \times 3^1 = 48$
$\mathbf{3^2}$	$3^2 = 9$	$2^1 \times 3^2 = 18$	$2^2 \times 3^2 = 36$	$2^3 \times 3^2 = 72$	$2^4 \times 3^2 = 144$

This type of table will work for any number whose prime decomposition contains exactly two different primes.

Since you have the option of using either none, one, two, three or four of the 2s, that gives 5 options. Equivalently, the 3^2 gives three options. As the table shows, multiplying these give all possible combinations – there are 5×3 of them: 15 factors.

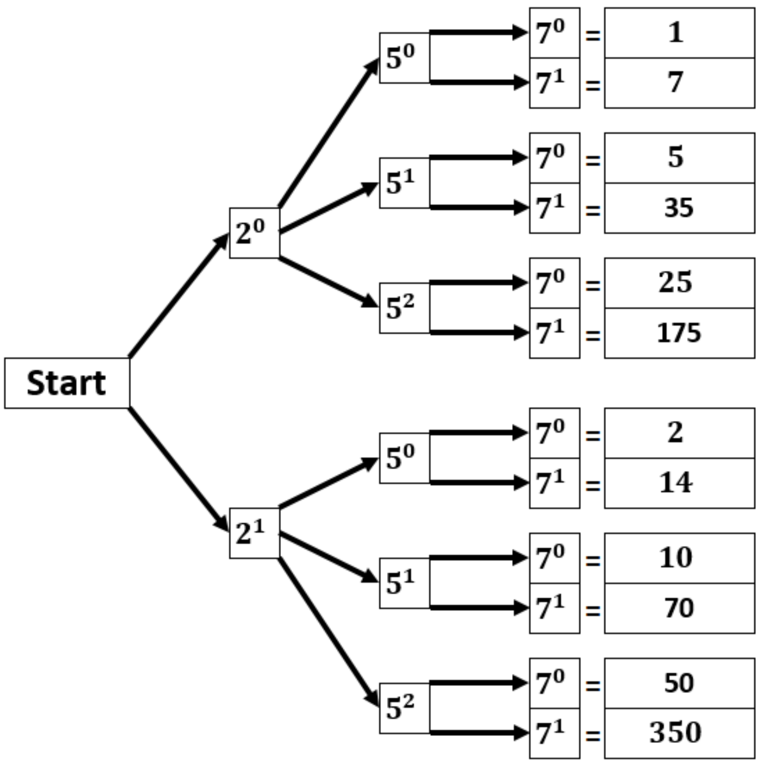
Conjecture 4: The total number of factors is equal to the product of the numbers one greater than the powers of primes in the decomposition.

More algebraically, if a number N is of the form: $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1, p_2, \dots p_k$ are distinct primes, the number of factors of N is given by: $(a_1 + 1)(a_2 + 1) \dots (a_k + 1)$.

Example: 360 has 24 factors, since $360 = 2^3 \times 3^2 \times 5$ and $(3 + 1)(2 + 1)(1 + 1) = 24$

Proof: Every factor of N must be a product of the same primes N is composed of, with powers no greater than those in N. Since not every prime need be included, a power of 0 is also acceptable. This means that a prime in the original number N with power x corresponds to $x + 1$ different choices for a factor of N. Given that these choices can be made independently for each prime number in N, the total number of choices must be the product of these values.

Put another way, to form a factor of, say 350, which can be written as $2 \times 5^2 \times 7$, you can choose to either have 2^0 (no 2s), or 2^1. This gives two options. Regardless of your choice, you can then decide to have no 5s (5^0), one 5 (5^1) or two 5s (5^2). So far this gives 2 choices followed by 3 choices (that's 6 options). Finally, you can either include a 7 (7^1) or not (7^0). These two choices double our total options, bringing us to 12. This can be considered in the form of a tree diagram:



Solutions to the challenge questions:

- Find the smallest number with exactly 7 factors.

7 factors means the numbers formed by adding 1 to each power of the prime decomposition multiply to make 7. Since 7 is prime, there is only one way to form it: 1×7. This means only one prime can be involved. Another way to think of it is that incorporating a single new prime to a factorisation will double the number of factors. 7 is odd, so cannot have been doubled. Any prime p will do: p^6 has 7 factors. The smallest possible is, therefore, 2^6, which is 64.

- Find the smallest number with exactly 17 factors.

Similar logic means we are looking for a number of the form p^{16}. The smallest such is 2^{16} which is 65,536. Note that this number can be written as $2^{2^{2^2}}$.

- Find all the numbers composed exclusively of single-digit primes with exactly 10 factors.

The only way for a number to have exactly 10 factors is to have primes with powers of one less than 1 and 10 (so, 0 and 9 – that is p^9 for any prime p) or one less than 2 and 5 (which would be 1 and 4 – that is, $p^4 \times q$ for primes p and q). Since we are limited to single-digit primes (otherwise there would be no end to the numbers we could list), we have:

$2^9, 3^9, 5^9, 7^9$
$2^4 \times 3, 2^4 \times 5, 2^4 \times 7$
$2 \times 3^4, 3^4 \times 5, 3^4 \times 7$
$2 \times 5^4, 3 \times 5^4, 5^4 \times 7$
$2 \times 7^4, 3 \times 7^4, 5 \times 7^4$ The smallest of these is $2^4 \times 3 = 48$, and the largest is $7^9 = 40{,}353{,}607$.

- Find the smallest number with exactly 100 factors.

One easy, but unlikely, candidate, is 2^{99}. It certainly has 100 factors, but unfortunately is 30 digits long.
To get the best result, we need low powers on our prime numbers. 100 itself is $2^2 \times 5^2$, so the most it will split up is into 2, 2, 5 and 5, giving powers of 1, 1, 4 and 4. Using these with the four lowest primes gives:

$$2^4 \times 3^4 \times 5 \times 7 = 45360$$

The next best candidates would probably be some version using three primes (but with necessarily higher powers) such as $2^4 \times 3^4 \times 5^3 = 162{,}000$.
Or the slightly larger, but pleasantly easy to factorise, $2^4 \times 3^3 \times 5^4 = 270000$.

- Describe the type of numbers which have a *prime* number of factors.

As described in the first bullet point, numbers of the form p^{q-1} where p is prime and q is prime.

- How many factors does 10! have? Remember $10! = 10 \times 9 \times 8 \times \ldots \times 3 \times 2 \times 1$.

For this it is only necessary to perform prime factor decomposition. Do each number separately at first:

$$2 \times 5 \times 3^2 \times 2^3 \times 7 \times 2 \times 3 \times 5 \times 2^2 \times 3 \times 2 = 2^8 \times 3^4 \times 5^2 \times 7$$
$$\Rightarrow \quad number\ of\ factors = 9 \times 5 \times 3 \times 2 = 270.$$ Note that $10! = 3{,}628{,}800$.

- How many factors does $7^6 + 7^7 + 7^8 + 7^9$ have?

Taking out a common factor of $7^6(1 + 7 + 7^2 + 7^3) = 7^6(1 + 7 + 49 + 343) = 7^6(400) = 2^4 \times 5^2 \times 7^6$

Now it is written as a product of primes we can use the rule: $(4 + 1)(2 + 1)(6 + 1) = 105$ factors.

- How many factors does $4^7 \times 6^8 \times 12^6$ have?

Writing this as a product of primes: $(2^2)^7 \times (2 \times 3)^8 \times (2^2 \times 3)^6 = 2^{14} \times 2^8 \times 3^8 \times 2^{12} \times 3^6 = 2^{34} \times 3^{14}$
Therefore $(34 + 1)(14 + 1) = 35 \times 15 = 525$ factors.