Deriving Circular Motion Formulae: Constant Angular Velocity

$$\boldsymbol{v = r\omega \quad a = r\omega^2 \quad a = \frac{v^2}{r}}$$

Stated assumptions:

$$v = \frac{dx}{dt} \tag{1}$$

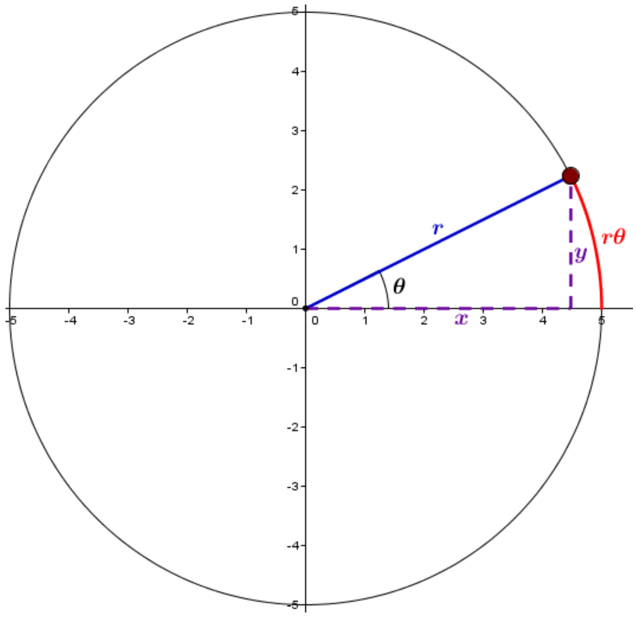
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2}$$

$$\omega = \frac{\theta}{t} \tag{3}$$

Defining variables:

$$\begin{aligned} r &= radius\ (m) \\ \theta &= angle\ (rad) \\ v &= velocity\ (ms^{-1}) \\ \omega &= angular\ speed\ (rad\ s^{-1}) \\ a &= acceleration\ (ms^{-2}) \\ t &= time\ (s) \end{aligned} \tag{4}$$

Deriving displacement in terms of r and ω:



When at an angle of θ around the circle of radius r, the position $\boldsymbol{x}$ is:

$$\boldsymbol{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} r\cos\omega t \\ r\sin\omega t \end{bmatrix} \tag{5}$$

(Using result (3) to write in terms of ω)

Note that the magnitude of the displacement is:

$$|\boldsymbol{x}| = \sqrt{r^2\cos^2\omega t + r^2\sin^2\omega t} = r\sqrt{\sin^2\omega t + \cos^2\omega t} = r$$

That is, the distance from the origin is fixed, and is equal to the radius.

Deriving velocity in terms of r and ω:

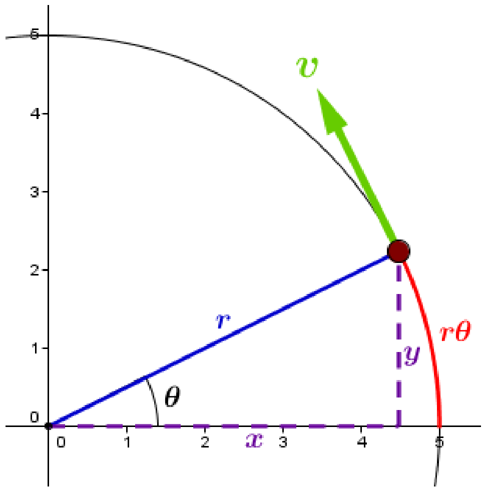
Applying (1) to result (5):

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} = \frac{d}{dt}\left(\begin{bmatrix} r\cos\omega t \\ r\sin\omega t \end{bmatrix}\right) = \begin{bmatrix} -\omega r\sin\omega t \\ \omega r\cos\omega t \end{bmatrix} \tag{6}$$

Note that the magnitude of the velocity is:

$$|\boldsymbol{v}| = \sqrt{(-\omega r\sin\omega t)^2 + (\omega r\cos\omega t)^2} = \omega r$$

And the direction of the velocity is given by $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ as shown:



That is, always perpendicular to the radial direction.

Deriving acceleration in terms of r and ω:

Applying (2) to result (6):

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left(\begin{bmatrix} -\omega r\sin\omega t \\ \omega r\cos\omega t \end{bmatrix}\right) = \begin{bmatrix} -\omega^2 r\cos\omega t \\ -\omega^2 r\sin\omega t \end{bmatrix}$$

$$\boldsymbol{a} = r\omega^2\begin{bmatrix} -\cos\omega t \\ -\sin\omega t \end{bmatrix} \tag{7}$$

Note: this is known as centripetal (centre-seeking) acceleration, acting in the exact opposite direction to the displacement vector (that is, radially towards the origin). The magnitude of centripetal acceleration is $r\omega^2$.

$$\boldsymbol{x = r}$$

$$\boldsymbol{v = r\omega,\ acting\ on\ a\ tangent\ to\ the\ circle}$$

$$\boldsymbol{a = r\omega^2 = \frac{v^2}{r},\ acting\ radially\ towards\ the\ centre}$$

Note that this proof relies on a constant angular speed. For a full proof, taking into account the potential for a variable angular speed (and therefore ending up with a transverse acceleration term), see below.

Deriving Circular Motion Formulae: Variable Angular Velocity

$$\boldsymbol{v = r\omega \quad a = r\omega^2 \quad a = \frac{v^2}{r}}$$

Stated assumptions:

$$v = \frac{dx}{dt} \tag{1}$$

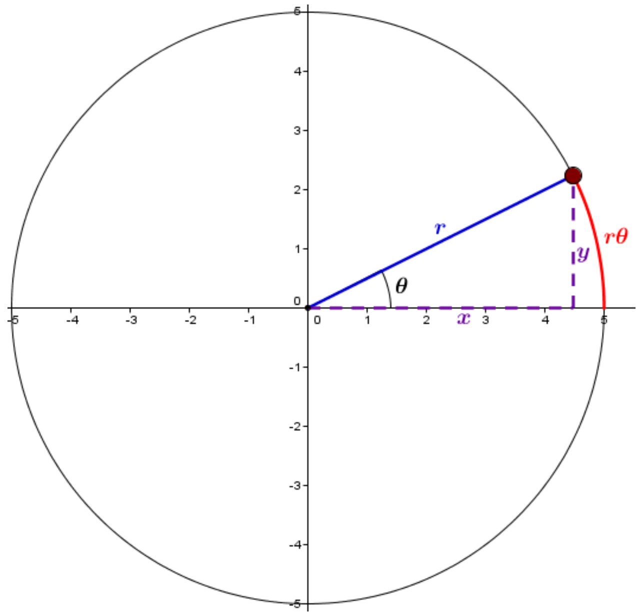
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2}$$

$$\omega = \frac{d\theta}{dt} \tag{3}$$

Defining variables:

$$\begin{aligned} &r = radius\ (m) \\ &\theta = angle\ (rad) \\ &v = velocity\ (ms^{-1}) \\ &\omega = angular\ speed\ (rad\ s^{-1}) \\ &a = acceleration\ (ms^{-2}) \\ &t = time\ (s) \end{aligned} \tag{4}$$

Deriving displacement in terms of r and θ:



When at an angle of θ around the circle of radius r, the position $\boldsymbol{x}$ is:

$$\boldsymbol{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \tag{5}$$

Note that the magnitude of the displacement is:

$$|\boldsymbol{x}| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{\sin^2\theta + \cos^2\theta} = r$$

That is, the distance from the origin is fixed, and is equal to the radius.

Deriving velocity in terms of r and ω:

Applying (1) to result (5):

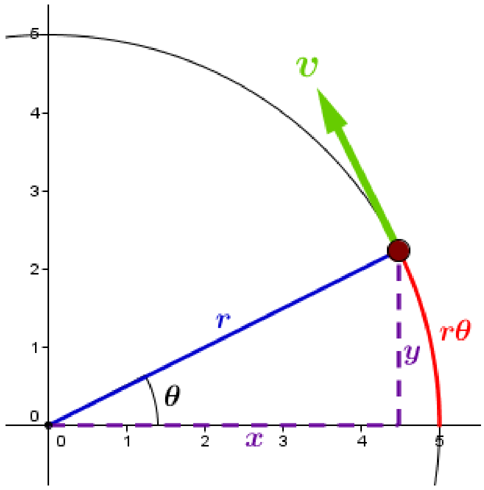
$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} = \frac{d}{dt}\left(\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) = \begin{bmatrix} -r\sin\theta\,\dfrac{d\theta}{dt} \\ r\cos\theta\,\dfrac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} -\omega r\sin\theta \\ \omega r\cos\theta \end{bmatrix} \tag{6}$$

Note that r is a constant for a given circle, but θ varies with time. Therefore we used implicit differentiation, and then applied formula (3) to convert to a form in terms of ω.

Note that the magnitude of the velocity is:

$$|\boldsymbol{v}| = \sqrt{(-\omega r\sin\theta)^2 + (\omega r\cos\theta)^2} = \omega r$$

And the direction of the velocity is given by $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ as shown:



That is, always perpendicular to the radial direction.

Deriving acceleration in terms of r and ω:

Applying (2) to result (6):

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left(\begin{bmatrix} -r\sin\theta\,\dfrac{d\theta}{dt} \\ r\cos\theta\,\dfrac{d\theta}{dt} \end{bmatrix}\right) = \begin{bmatrix} -r\left(\sin\theta\,\dfrac{d^2\theta}{dt^2} + \cos\theta\,\dfrac{d\theta}{dt}\dfrac{d\theta}{dt}\right) \\ r\left(\cos\theta\,\dfrac{d^2\theta}{dt^2} - \sin\theta\,\dfrac{d\theta}{dt}\dfrac{d\theta}{dt}\right) \end{bmatrix} \tag{7}$$

$$\boldsymbol{a} = r\frac{d\omega}{dt}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} + r\omega^2\begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}$$

Note: this expression has two distinct components, and writing them separated like this enables us to see what they represent.
The first part represents the transverse acceleration (in the direction of the velocity), and is related to the rate of change of the angular velocity (if ω is constant, this term disappears).
The second term is known as the centripetal (centre-seeking) acceleration, acting in the exact opposite direction to the displacement vector (that is, radially towards the origin). The magnitude of centripetal acceleration is $r\omega^2$.

The above proof still makes some assumptions (valid for circular motion) about the rate of change of the radius (namely, that it is constant). For the even more general scenario (eg motion in a spiral, etc), you will need to look into polar coordinates: http://thechalkface.net/resources/polar_coordinates.pdf.