Dealing with a quadratic

What it is:

A quadratic expression is an algebraic expression containing an x^2 term, as well as possibly an x term and/or a number, but nothing else - eg, no x^3 term.

The general form of a simplified quadratic expression is $ax^2 + bx + c$ where a, b and c are numbers ($a \neq 0$ as $a = 0$ would give a linear expression).

What you may need to do:

1. Formulate a quadratic expression or equation.
2. Solve a quadratic equation.
3. Complete the square to identify a maximum or minimum point.
4. Draw the graph of a quadratic function.
5. Simplify, add or subtract rational expressions (fractions involving quadratic expressions).

1. Formulate a quadratic expression or equation.

Generate a quadratic expression from a description or diagram.

This will generally involve interpreting the description or diagram algebraically, using a rule or formula (eg volume of a prism, Pythagoras) to connect the different elements, and simplifying the algebraic expression you produce by multiplying out brackets and collecting like terms.

Eg:
One of the two shortest sides of a right-angled triangle is $3cm$ lower than the other.
Show that the *square* of the hypotenuse is equal to $2x^2 + 6x + 9$, where x is the length of the shortest side.

$$x^2 + (x+3)^2 = c^2 \quad \Longrightarrow \quad c^2 = x^2 + (x^2 + 6x + 9) = \mathbf{2x^2 + 6x + 9}$$

2. Solve a quadratic equation.

Use one of the following methods to determine values for x which will make the equation true. Note that some quadratics have no solutions, some have just one and some have two. Sometimes even if an equation has solutions, they cannot be found by factorising since they are not 'nice' numbers (eg $x = \frac{3+\sqrt{2}}{5}$).

Simple rearranging:

This is only possible if x occurs in just one position in the equation. In this case, rearrange the equation as you normally would (remembering that square-rooting gives two possible solutions) until you can write as $x = \cdots$.

Eg:

Solve: $2x^2 - 18 = 0$

$$2x^2 = 18 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad \boldsymbol{x = \pm 3}$$

Eg 2:

Solve $4(x-3)^2 - 25 = 0$

$$4(x-3)^2 = 25 \quad \Rightarrow \quad (x-3)^2 = \frac{25}{4} \quad \Rightarrow \quad x - 3 = \pm\frac{5}{2}$$
$$\Rightarrow \quad x = 3 \pm \frac{5}{2} \quad \Rightarrow \quad \boldsymbol{x = 5.5 \quad or \quad x = 0.5}$$

Note: this second example has the quadratic in completed square form. It is possible to rearrange any quadratic expression into this form – this will be looked at in detail later.

Single term factorising:

This is only possible if there are no number terms in the quadratic (that is, just x^2 and x terms). Then it is possible to take out a common factor of x from both terms.

Eg:

Solve: $3x^2 - 2x = 0$

$$3x^2 - 2x = 0 \quad \Rightarrow \quad x(3x-2) = 0 \quad \Rightarrow \quad \boldsymbol{x = 0 \quad or \quad x = \frac{2}{3}}$$

Note: when a quadratic expression is set equal to zero, if it is fully factorised, solutions can be found by determining the values of x for which any factor equals zero.

Double bracket factorising:
This is only possible if the solutions for x are 'nice' numbers. If the solutions are surds, for instance, it will not be at all straightforward to factorise even though there are solutions. Use either completing the square or the formula in these cases. For quadratics that do factorise, there are two main types:

Factorising expressions with a single x^2 term:
Write out two brackets with x at the start of each. To determine the numbers to go alongside these, find two numbers that multiply to make the number but add to make the x coefficient.

Eg:
Solve $x^2 - 5x + 4 = 0$

$x^2 - 5x + 4 = (x \ldots)(x \ldots)$ and since -1 and -4 multiply to make 4 but add to make -5, $x^2 - 5x + 4 = 0 \quad \Longrightarrow \quad (x-1)(x-4) = 0 \quad \Longrightarrow \quad \boldsymbol{x = 1 \quad or \quad x = 4}$

Factorising expressions with a number in front of x^2:
Before splitting into brackets, find two numbers that multiply to make the product of the x^2 coefficient and the number, but add to make the x coefficient. Then split the x term into these two parts, factorise the first two terms of the expression separately, then the last two terms, and finally factorise the whole thing.

Eg:
Solve: $2x^2 + x - 3 = 0$

Two numbers which multiply to make $2 \times -3 = -6$ and add to make 1: 3 and -2:

$$2x^2 + x - 3 = 2x^2 + 3x - 2x - 3 = x(2x+3) - 1(2x+3) = (2x+3)(x-1)$$

$$\Longrightarrow \quad (2x+3)(x-1) = 0 \quad \Longrightarrow \quad \boldsymbol{x = -\frac{3}{2} \quad or \quad x = 1}$$

Using the formula:
This works for any quadratic, but can take longer than factorising for simpler equations. Quickest with a calculator, but can be used without a calculator to give exact answers (eg, in surd form). Make sure the equation is written in the form $ax^2 + bx + c = 0$, then substitute your values for a, b and c into the formula (given in the front of the GCSE paper).

Eg:
Solve $3x^2 - x - 12 = 0$ giving your answers to 2 d.p.

$a = 3$, $b = -1$, $c = -12$, so:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Longrightarrow \quad x = \frac{1 \pm \sqrt{1 + 144}}{6} \quad \Longrightarrow \quad \boldsymbol{x = 2.17 \quad or \quad x = -1.84}$$

Completing the square:

This works for any quadratic, but can take longer than factorising for simpler equations. Note also that completing the square is a technique that can be used to determine a maximum or minimum point on a quadratic graph without solving the equation. The aim is to write the quadratic expression in the form $p(x+q)^2+r$, as shown below.

Method:

Write the following expression in completed square form: $x^2+6x+10$

Step 1:

Halve the x coefficient to find the number to go with x in the squared bracket:

$$(x+3)^2$$

We know that if we multiply this out we would get x^2+6x+9

Step 2:

Subtract the square of this number from the squared bracket:

$$(x+3)^2-9$$

By subtracting the 9 we turn the expression into x^2+6x

Step 3:

Add on the number part from the original expression, and simplify:

$$(x+3)^2-9+10=\boldsymbol{(x+3)^2+1}$$

This is the same as $x^2+6x+10$*, but written in completed square form*

We can then solve the equation by rearranging (notice that now x occurs just once):

Eg:

Solve $x^2+6x-5=0$ by completing the square.

$$x^2+6x-5=(x+3)^2-9+1=(x+3)^2-8 \quad \Rightarrow \quad (x+3)^2-8=0$$
$$\Rightarrow \quad (x+3)^2=8 \quad \Rightarrow \quad x+3=\pm\sqrt{8}=\pm2\sqrt{2} \quad \Rightarrow \quad \boldsymbol{x=-3\pm2\sqrt{2}}$$

3. Complete the square to identify a maximum or minimum point.

Sometimes we use completing the square not to solve a quadratic equation but to better describe a quadratic expression. In this case, always note that the squared bracket is never less than 0*.*

By rewriting a quadratic expression in completed square form, the lowest possible value can be determined (or greatest possible if a negative quadratic graph). This is done by setting the squared bracket equal to 0.

Eg:
Find the minimum value of $x^2 + 8x + 10$.

$$x^2 + 8x + 10 = (x+4)^2 - 16 + 10 = (x+4)^2 - 6$$
$$minimum\ when\ (x+4)^2 = 0 \quad \Rightarrow \quad \boldsymbol{min = -6}$$

Note: Anything squared is non-negative (positive or zero). If you have completed the square with a negative quadratic the squared bracket will have a minus sign beside it, making the final result a maximum rather than a negative.

Eg:
Find the maximum of $-x^2 + 3x + 2$.

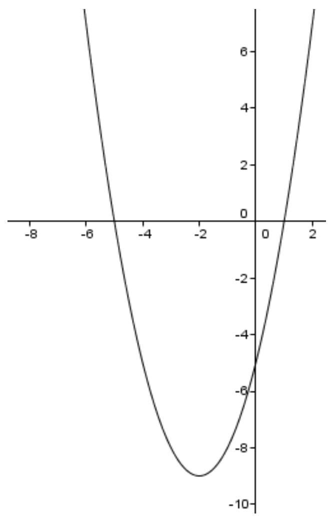
$$-x^2 + 3x + 2 = -[x^2 - 3x - 2] = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 2\right] = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{17}{4}\right]$$
$$= -\left(x - \frac{3}{2}\right)^2 + \frac{17}{4} \quad which\ has\ a\ maximum\ when\ \left(x - \frac{3}{2}\right)^2 = 0 \quad \Rightarrow \quad \boldsymbol{max = \frac{17}{4}}$$

4. Draw the graph of a quadratic function.

A quadratic graph function may be given in the form $y = ax^2 + bx + c$*. The points where the graph crosses the* x*-axis are the solutions for* x *when* $y = 0$*, and the point where it crosses the* y*-axis is the solution for* y *where* $x = 0$*. It may be necessary to interpret a quadratic function involving the max/min point found by completing the square.*

Eg:
Sketch the graph of $y = (x+2)^2 - 9$.

Setting $x = 0$: $y = 2^2 - 9 = -5 \quad \Rightarrow \quad \boldsymbol{(0, -5)}$
Setting $y = 0$: $0 = (x+2)^2 - 9$
$\Rightarrow \quad (x+2)^2 = 9 \quad \Rightarrow \quad x = -2 \pm 3 \quad \Rightarrow \quad \boldsymbol{(1, 0), (-5, 0)}$
Setting $(x+2)^2 = 0$: $y = -9 \quad \Rightarrow$ **min:** $\boldsymbol{(-2, -9)}$



5. Simplify, add or subtract rational expressions.

The rules of fraction still apply to algebraic fractions. You can cancel terms if they are factors of both the numerator and denominator (that is, divide top and bottom by the same thing), and to add or subtract fractions, first find a common denominator then combine the numerators by adding or subtracting.

Eg:

Simplify $\frac{x^2-100}{x^2+8x-20}$.

$$\frac{x^2-100}{x^2+8x-20} = \frac{(x+10)(x-10)}{(x+10)(x-2)} = \frac{\boldsymbol{x-10}}{\boldsymbol{x-2}}$$

Eg 2:

Simplify $\frac{x+4}{2x} - \frac{x-8}{x^2-4x}$.

$$\frac{x+4}{2x} - \frac{x-8}{x^2-4x} = \frac{x+4}{2x} - \frac{x-8}{x(x-4)} = \frac{(x+4)(x-4)}{2x(x-4)} - \frac{2(x-8)}{2x(x-4)}$$

$$= \frac{(x+4)(x-4)-2(x-8)}{2x(x-3)} = \frac{x^2-16-2x+16}{2x(x-3)}$$

$$= \frac{x^2-2x}{2x(x-3)} = \frac{x(x-2)}{2x(x-3)} = \frac{\boldsymbol{x-2}}{\boldsymbol{2(x-3)}}$$

Hard Exam Question

A bag contains $(n + 7)$ tennis balls. n of the balls are yellow. The other 7 balls are white. John will take at random a ball from the bag. He will look at its colour and then put it back in the bag.

a)

i.

Write down an expression, in terms of n, for the probability that John will take a white ball.

Bill states that the probability that John will take a white ball is $\frac{2}{5}$.

ii.

Prove that Bill's statement cannot be correct.

After John has put the ball back into the bag, Mary will then take at random at ball from the bag. She will note its colour.

b)

Given that the probability that John and Mary will take balls with different colours is $\frac{4}{9}$, prove that $2n^2 - 35n + 98 = 0$.

c)

i.

Factorise the expression $2n^2 - 35n + 98$.

ii.

Hence solve the equation $2n^2 - 35n + 98 = 0$.

d)

Using your answer to part c)ii or otherwise, calculate the probability that John and Mary will both take white balls.

(see next page for solutions)

Hard Exam Question Solutions

A bag contains $(n+7)$ tennis balls. n of the balls are yellow. The other 7 balls are white. John will take at random a ball from the bag. He will look at its colour and then put it back in the bag.

a)

i.

Write down an expression, in terms of n, for the probability that John will take a white ball.

7 out of $n+7$ balls are white, so the probability is $\frac{7}{n+7}$.

Bill states that the probability that John will take a white ball is $\frac{2}{5}$.

ii.

Prove that Bill's statement cannot be correct.

Assume $\frac{7}{n+7}=\frac{2}{5}$. Then $7=\frac{2(n+7)}{5} \quad \Rightarrow \quad 35=2(n+7) \quad \Rightarrow \quad n=10.5$. Not a whole number, so cannot be a number of balls.

After John has put the ball back into the bag, Mary will then take at random at ball from the bag. She will note its colour.

b)

Given that the probability that John and Mary will take balls with different colours is $\frac{4}{9}$, prove that $2n^2-35n+98=0$.

White then yellow or yellow then white: $\frac{7}{n+7}\times\frac{n}{n+7}+\frac{n}{n+7}\times\frac{7}{n+7}=2\left(\frac{7n}{(n+7)^2}\right)=\frac{14n}{n^2+14n+49}$

$$\frac{14n}{n^2+14n+49}=\frac{4}{9} \quad \Rightarrow \quad \frac{63n}{n^2+14n+49}=2 \quad \Rightarrow \quad 63n=2n^2+28n+98$$
$$\Rightarrow \quad 2n^2-35n+98=0 \ as\ required$$

c)

i.

Factorise the expression $2n^2-35n+98$.

Two numbers that multiply to make 196 but add to make -35: -7 and -28.

$$2n^2-7n-28n+98=n(2n-7)-14(2n-7)=(2n-7)(n-14)$$

ii.

Hence solve the equation $2n^2-35n+98=0$.

$$(2n-7)(n-14)=0 \quad \Rightarrow \quad n=\frac{7}{2} \quad or \quad n=14$$

d)

Using your answer to part c)ii or otherwise, calculate the probability that John and Mary will both take white balls.

White then white, with $n=14$ (since n cannot equal 3.5): $P(white)=\frac{7}{21}=\frac{1}{3}$

Probability that both are white: $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$