VW Beetle beats Porsche 911 in 1 mile race!

In an unprecedented race conducted by the renowned Top Gear team, a VW Beetle (circa 1967) gets the jump on a Porsche 911 Turbo.

To even up the odds, while the Porsche is driven along a standard mile-long desert track, the Beetle gets a helping hand from gravity by being dropped by helicopter, 1 mile above the finish line.

The Porsche should be able to complete a drag race such as this in around 37 seconds. Assuming air resistance is negligible, calculate the time taken for the Beetle to reach the finish line. Use $1\,mile \approx 1600\,metres$.

Using energy considerations, calculate the maximum speed of the Beetle.

In practice, the top speed of the 840kg Beetle was $49.7ms^{-1}$ (roughly 110mph). Calculate the work done against resistive forces during the motion.

A better model of the situation takes into account air resistance, which is given by: $R = kv^2$, where R is the force of air resistance and v is the velocity. After 8 seconds, the Beetle is experiencing an acceleration of $1.46ms^{-2}$ and is travelling at a speed of $45ms^{-2}$. Show that $k = 3.46 \; to \; 3 \; s.f.$

It can be shown that the velocity of a body of mass m, dropped from rest and falling under gravity, which experiences a resistance force $R = kv^2$ is given by:

$$v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right)$$

Use the standard result $\int \tanh x \; dx = \ln(\cosh x)$ to find an expression for the displacement of such a body at time t, and hence calculate the time taken for the Beetle to reach the ground.

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SOLUTIONS

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$$s = 1600 \quad u = 0 \quad v = - \quad a = 9.8 \quad t = ?$$

$$s = ut + \frac{1}{2}at^2 \quad \Longrightarrow \quad 1600 = 4.9t^2 \quad \Longrightarrow \boldsymbol{t = 18.1s\ to\ 3\ s.f.}$$

Using energy considerations, calculate the maximum speed of the Beetle.

$$All\ GPE\ is\ converted\ to\ KE:$$

$$mgh = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{2 \times 1600 \times 9.8} = \boldsymbol{177ms^{-1}} \approx 400mph$$

In practice, the top speed of the 840kg Beetle was $49.7ms^{-1}$ (roughly 110mph). Calculate the work done against resistive forces during the motion.

$$Initial\ energy = GPE = 9.8 \times 1600 \times 840 = 13{,}171{,}200J$$

$$Final\ energy = KE = \frac{1}{2}(49.7^2) \times 840 = 1{,}037{,}437.8J$$

$$Lost\ to\ resistive\ forces = 13{,}171{,}200 - 1{,}037{,}437.8 = \boldsymbol{12{,}133{,}762{,}2J}$$

A better model of the situation takes into account air resistance, which is given by: $R = kv^2$, where R is the force of air resistance and v is the velocity. After 8 seconds, the Beetle is experiencing an acceleration of $1.46ms^{-2}$ and is travelling at a speed of $45ms^{-2}$. Show that $k = 3.46 \; to \; 3 \; s.f.$

$$Resultant\ force = 840g - kv^2$$

$$840a = 840g - kv^2 \quad \Rightarrow \quad a = g - \frac{kv^2}{840}$$

$$a = 1.46 \; when \; v = 45 \quad \Rightarrow \quad 1.46 = g - \frac{k(45^2)}{840} \quad \Rightarrow \quad = 3.46 \; to \; 3 \; s.f.$$

It can be shown that the velocity of a body of mass m, dropped from rest and falling under gravity, which experiences a resistance force $R = kv^2$ is given by:

$$v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right)$$

Use the standard result $\int \tanh x \; dx = \ln(\cosh x)$ to find an expression for the displacement of such a body at time t, and hence calculate the time taken for the Beetle to reach the ground.

$$x = \int v \, dt = \int \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) dt = \sqrt{\frac{mg}{k}} \int \tanh\left(\sqrt{\frac{kg}{m}} t\right) dt$$

$$\Rightarrow \quad x = \sqrt{\frac{mg}{k}} \left[\sqrt{\frac{m}{kg}} \ln\left(\cosh \sqrt{\frac{kg}{m}} t\right) \right] + C \quad (since \; x = 0 \; at \; t = 0, C = 0)$$

$$x = \frac{m}{k} \ln\left(\cosh \sqrt{\frac{kg}{m}} t\right) \quad \Rightarrow \quad t = \sqrt{\frac{m}{kg}} \cosh^{-1} e^{\frac{kx}{m}}$$

$$For\ the\ Beetle: \quad t = \sqrt{\frac{840}{3.46 \times 9.8}} \cosh^{-1} e^{\frac{3.46 \times 1600}{840}} = \mathbf{36.3s \; to \; 3 \; s.f.}$$