General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Monday 19 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Use the definitions $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ to show that

$$1 + 2\sinh^2\theta = \cosh 2\theta$$ *(3 marks)*

(b) Solve the equation

$$3\cosh 2\theta = 2\sinh\theta + 11$$

giving each of your answers in the form $\ln p$. *(6 marks)*

2 (a) Indicate on an Argand diagram the region for which $|z - 4i| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4i| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

3 (a) Given that $f(r) = \frac{1}{4}r^2(r+1)^2$, show that

$$f(r) - f(r-1) = r^3$$ *(3 marks)*

(b) Use the method of differences to show that

$$\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$$ *(5 marks)*

4 It is given that α, β and γ satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha^2 + \beta^2 + \gamma^2 &= -5 \\ \alpha^3 + \beta^3 + \gamma^3 &= -23 \end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

5 (a) Given that $u = \cosh^2 x$, show that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sinh 2x$. *(2 marks)*

(b) Hence show that

$$\int_0^1 \frac{\sinh 2x}{1 + \cosh^4 x}\,\mathrm{d}x = \tan^{-1}(\cosh^2 1) - \frac{\pi}{4}$$ *(5 marks)*

6 Prove by induction that

$$\frac{2 \times 1}{2 \times 3} + \frac{2^2 \times 2}{3 \times 4} + \frac{2^3 \times 3}{4 \times 5} + \ldots + \frac{2^n \times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for all integers $n \geqslant 1$. *(7 marks)*

Turn over for the next question

7 (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sqrt{1-x^2}}{x}$; *(4 marks)*

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ is $\ln 3$. *(5 marks)*

8 (a) Show that

$$(z^4 - \mathrm{e}^{\mathrm{i}\theta})(z^4 - \mathrm{e}^{-\mathrm{i}\theta}) = z^8 - 2z^4\cos\theta + 1$$ *(2 marks)*

(b) Hence solve the equation

$$z^8 - z^4 + 1 = 0$$

giving your answers in the form $\mathrm{e}^{\mathrm{i}\phi}$, where $-\pi < \phi \leqslant \pi$. *(6 marks)*

(c) Indicate the roots on an Argand diagram. *(3 marks)*

END OF QUESTIONS