General Certificate of Education
January 2006
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 27 January 2006 1.30 pm to 3.00 pm

For this paper you must have: • an 8-page answer book • the **blue** AQA booklet of formulae and statistical tables You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions

- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information

- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice

- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 (a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots$$ *(4 marks)*

2 The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

3 The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \qquad \text{and} \qquad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1 = \mathrm{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1)$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4\left(\sqrt{2} + 1\right)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

Turn over for the next question

6 It is given that $z = e^{i\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + ib$. *(5 marks)*

7 (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

END OF QUESTIONS