AQA MPC4 Core 4 Mathematics 12 June 2014

Question Paper and Worked Solutions

Please note, this document represents my own solutions to the questions, is entirely unofficial and is not related to the mark scheme (which I have not seen). Therefore, while it should help you see how to do the questions, it won't include every valid method or give you a break down of the mark allocation. If you spot any errors, or think you have found a better solution, please email me so I can update it.

1 A curve is defined by the parametric equations $x = \frac{t^2}{2} + 1$, $y = \frac{4}{t} - 1$.

(a) Find the gradient at the point on the curve where $t = 2$.

[3 marks]

(b) Find a Cartesian equation of the curve.

[2 marks]

1.

a)

$$\frac{dx}{dt} = t \qquad \frac{dy}{dt} = -4t^{-2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{4t^{-2}}{t} = -4t^{-3} = -\frac{4}{t^3} \qquad t = 2 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{4}{2^3} = -\mathbf{\frac{1}{2}}$$

b)

$$y = \frac{4}{t} - 1 \quad \Rightarrow \quad y + 1 = \frac{4}{t} \quad \Rightarrow \quad t = \frac{4}{y+1}$$

$$x = \frac{t^2}{2} + 1 = \frac{\left(\frac{4}{y+1}\right)^2}{2} + 1 \quad \Rightarrow \quad \boldsymbol{x = \frac{8}{(y+1)^2} + 1}$$

2 (a) Given that $\dfrac{4x^3-2x^2+16x-3}{2x^2-x+2}$ can be expressed as $Ax+\dfrac{B(4x-1)}{2x^2-x+2}$, find the values of the constants A and B.

[3 marks]

(b) The gradient of a curve is given by

$$\frac{dy}{dx}=\frac{4x^3-2x^2+16x-3}{2x^2-x+2}$$

The point $(-1, 2)$ lies on the curve. Find the equation of the curve.

[4 marks]

2.

a)

$$Ax+\frac{B(4x-1)}{2x^2-x+2}=\frac{Ax(2x^2-x+2)+B(4x-1)}{2x^2-x+2}=\frac{4x^3-2x^2+16x-3}{2x^2-x+2}$$

$$\Rightarrow \quad Ax(2x^2-x+2)+B(4x-1)=4x^3-2x^2+16x-3$$

$$\Rightarrow \quad 2Ax^3-Ax^2+2Ax+4Bx-B=4x^3-2x^2+16x-3$$

Comparing coefficients: $\boldsymbol{A=2}\quad and \quad \boldsymbol{B=3}$

b)

$$\int 1\,dy=\int\frac{4x^3-2x^2+16x-3}{2x^2-x+2}\,dx$$

$$\Rightarrow \quad y=\int 2x+\frac{3(4x-1)}{2x^2-x+2}\,dx=\int 2x\,dx+3\int\frac{4x-1}{2x^2-x+2}\,dx=x^2+3\ln(2x^2-x+2)+C$$

$$x=-1 \quad at \quad y=2 \quad \Rightarrow \quad 2=(-1)^2+3\ln(2(-1)^2-(-1)+2)+C$$

$$\Rightarrow \quad 2=1+3\ln 5+C \quad \Rightarrow \quad C=1-3\ln 5 \quad \Rightarrow \quad \boldsymbol{y=x^2+3\ln(2x^2-x+2)+1-3\ln 5}$$

3 (a) Find the binomial expansion of $(1-4x)^{\frac{1}{4}}$ up to and including the term in x^2.

[2 marks]

(b) Find the binomial expansion of $(2+3x)^{-3}$ up to and including the term in x^2.

[3 marks]

(c) Hence find the binomial expansion of $\dfrac{(1-4x)^{\frac{1}{4}}}{(2+3x)^3}$ up to and including the term in x^2.

[2 marks]

3.

a)

$$(1-4x)^{\frac{1}{4}}\approx 1+\left(\frac{1}{4}\right)(-4x)+\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\frac{(-4x)^2}{2}=\boldsymbol{1-x-\frac{3}{2}x^2}$$

b)

$$(2+3x)^{-3}\approx 2^{-3}\left(1+\frac{3}{2}x\right)^{-3}=\frac{1}{8}\left[1+(-3)\left(\frac{3}{2}x\right)+\frac{(-3)(-4)\left(\frac{3}{2}x\right)^2}{2}\right]=\boldsymbol{\frac{1}{8}-\frac{9}{16}x+\frac{27}{16}x^2}$$

c)

$$\frac{(1-4x)^{\frac{1}{4}}}{(2+3x)^3}=(1-4x)^{\frac{1}{4}}(2+3x)^{-3}=\left(1-x-\frac{3}{2}x^2\right)\left(\frac{1}{8}-\frac{9}{16}x+\frac{27}{16}x^2\right)\approx\boldsymbol{\frac{1}{8}-\frac{11}{16}x+\frac{33}{16}x^2}$$

4 A painting was valued on 1 April 2001 at £5000.

The value of this painting is modelled by

$$V = Ap^t$$

where £V is the value t years after 1 April 2001, and A and p are constants.

(a) Write down the value of A.

[1 mark]

(b) According to the model, the value of this painting on 1 April 2011 was £25 000.

Using this model:

(i) show that $p^{10} = 5$;

[1 mark]

(ii) use logarithms to find the year in which the painting will be valued at £75 000.

[4 marks]

(c) A painting by another artist was valued at £2500 on 1 April 1991. The value of this painting is modelled by

$$W = 2500q^t$$

where £W is the value t years after 1 April 1991, and q is a constant.

(i) Show that, according to the two models, the value of the two paintings will be the same T years after 1 April 1991,

$$\text{where } T = \frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{p}{q}\right)}$$

[4 marks]

(ii) Given that $p = 1.029q$, find the year in which the two paintings will have the same value.

[1 mark]

4.

a)

$$\boldsymbol{A = 5000}$$

b)

i.

$$V = 5000p^t \quad \Rightarrow \quad 25000 = 5000p^{10} \quad \Rightarrow \quad \boldsymbol{p^{10} = 5}$$

ii.

$$V = 5000 \times \left(5^{\frac{1}{10}}\right)^t = 5000 \times 5^{0.1t} \quad \Rightarrow \quad 75000 = 5000 \times 5^{0.1t} \quad \Rightarrow \quad 15 = 5^{0.1t} \quad \Rightarrow \quad \ln 15 = \ln 5^{0.1t}$$

$$\Rightarrow \quad \ln 15 = 0.1t \ln 5 \quad \Rightarrow \quad \frac{\ln 15}{0.1 \ln 5} = t \approx 16.83 \quad \Rightarrow \quad \textit{between } 2017 \textit{ and } 2018$$

$$1\,Jan\ 2018 \quad \Rightarrow \quad t = 16 + \frac{9}{12} = 16.75 \quad \textit{therefore value becomes } £75000 \textit{ in } \mathbf{2018}$$

c)
i.

$$5000p^{T-10} = 2500q^T \quad \Rightarrow \quad 2p^{T-10} = q^T \quad \Rightarrow \quad \ln(2p^{T-10}) = \ln(q^T) \quad \Rightarrow \quad \ln 2 + (T-10)\ln p = T \ln q$$

$$\Rightarrow \quad \ln 2 + T\ln p - 10\ln p - T\ln q = 0 \quad \Rightarrow \quad T(\ln p - \ln q) = 10\ln p - \ln 2 \quad \Rightarrow \quad T = \frac{10\ln p - \ln 2}{\ln p - \ln q}$$

$$T = \frac{\ln p^{10} - \ln 2}{\ln\left(\frac{p}{q}\right)} = \frac{\ln 5 - \ln 2}{\ln\left(\frac{p}{q}\right)} = \frac{\boldsymbol{\ln\left(\frac{5}{2}\right)}}{\boldsymbol{\ln\left(\frac{p}{q}\right)}}$$

ii.

$$p = 1.029q \quad \Rightarrow \quad T = \frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{1.029q}{q}\right)} = \frac{\ln\left(\frac{5}{2}\right)}{\ln(1.029)} \approx 32.052 \quad \Rightarrow \quad \boldsymbol{The\ year\ 2023}$$

5 (a) (i) Express $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0° < \alpha < 90°$, giving your value of α to the nearest $0.1°$.

[3 marks]

(ii) Hence solve the equation $3\sin 2\theta + 4\cos 2\theta = 5$ in the interval $0° < \theta < 360°$, giving your solutions to the nearest $0.1°$.

[3 marks]

(b) (i) Show that the equation $\tan 2\theta \tan\theta = 2$ can be written as $2\tan^2\theta = 1$.

[2 marks]

(ii) Hence solve the equation $\tan 2\theta \tan\theta = 2$ in the interval $0° \leqslant \theta \leqslant 180°$, giving your solutions to the nearest $0.1°$.

[2 marks]

(c) (i) Use the Factor Theorem to show that $2x - 1$ is a factor of $8x^3 - 4x + 1$.

[1 mark]

(ii) Show that $4\cos 2\theta\cos\theta + 1$ can be written as $8x^3 - 4x + 1$ where $x = \cos\theta$.

[1 mark]

(iii) Given that $\theta = 72°$ is a solution of $4\cos 2\theta\cos\theta + 1 = 0$, use the results from parts **(c)(i)** and **(c)(ii)** to show that the exact value of $\cos 72°$ is $\frac{(\sqrt{5}-1)}{p}$ where p is an integer.

[3 marks]

5.
a)
i.

$$3\sin x + 4\cos x = R\sin(x+\alpha) \quad \Rightarrow \quad R = \sqrt{3^2+4^2} = 5$$

$$\sin(x+\alpha) = \sin x\cos\alpha + \cos x\sin\alpha \quad \Rightarrow \quad 3\sin x + 4\cos x = 5\cos\alpha\sin x + 5\sin\alpha\cos x$$

$$\Rightarrow \quad 3 = 5\cos\alpha \quad and \quad 4 = 5\sin\alpha \quad \Rightarrow \quad \alpha = 53.1°\ to\ 1\ d.p. \quad \Rightarrow \quad 3\sin x + 4\cos x = \boldsymbol{5\sin(x + 53.1°)}$$

ii.

$$0° < \theta < 360° \quad \Rightarrow \quad 53.1° < 2\theta + 53.1° < 773.1°$$

$$3\sin 2\theta + 4\cos 2\theta = 5 \quad \Rightarrow \quad 5\sin(2\theta + 53.1°) = 5 \quad \Rightarrow \quad \sin(2\theta + 53.1°) = 1$$

$$\Rightarrow \quad 2\theta + 53.1° = 90°, 450° \quad \Rightarrow \quad \boldsymbol{\theta = 18.4°} \quad \boldsymbol{or} \quad \boldsymbol{\theta = 198.4°}$$

b)
i.

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad\Rightarrow\quad \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta\tan\theta=2 \quad\Rightarrow\quad \frac{2\tan^2\theta}{1-\tan^2\theta}=2 \quad\Rightarrow\quad 2\tan^2\theta=2-2\tan^2\theta \quad\Rightarrow\quad 4\tan^2\theta=2 \quad\Rightarrow\quad \mathbf{2\tan^2\theta=1}$$

ii.

$$0°\le\theta\le 180°$$

$$\tan 2\theta\tan\theta=2 \quad\Rightarrow\quad 2\tan^2\theta=1 \quad\Rightarrow\quad \tan\theta=\pm\frac{1}{\sqrt{2}} \quad\Rightarrow\quad \boldsymbol{\theta=35.3°, 144.7°}$$

c)
i.

$$(ax+b)\ is\ a\ factor\ of\ f(x) \quad\Leftrightarrow\quad f\left(-\frac{b}{a}\right)=0$$

$$f(x)=8x^3-4x+1 \qquad f\left(\frac{1}{2}\right)=8\left(\frac{1}{2}\right)^3-4\left(\frac{1}{2}\right)+1=1-2+1=0 \quad\Rightarrow\quad \boldsymbol{(2x-1)\ is\ a\ factor}$$

ii.

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \quad\Rightarrow\quad \cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$$

$$4\cos 2\theta\cos\theta+1=4(2\cos^2\theta-1)\cos\theta+1=\mathbf{8\cos^3\theta-4\cos\theta+1=8x^3-4x+1} \quad \boldsymbol{where\ x=\cos\theta}$$

iii.

$$8x^3-4x+1=(2x-1)(4x^2+2x-1) \quad\Rightarrow\quad x=\frac{1}{2} \quad\Rightarrow\quad \theta=\cos^{-1}\frac{1}{2}=60°, 300°, \ldots$$

$$OR:\quad \cos\theta=x=\frac{-2\pm\sqrt{2^2-4(4)(-1)}}{8}=\frac{-2\pm\sqrt{20}}{8}=\frac{-2\pm 2\sqrt{5}}{8}=\frac{-1+\sqrt{5}}{4} \quad or \quad \frac{-1-\sqrt{5}}{4}$$

$$\cos\theta=\frac{(-1-\sqrt{5})}{4} \quad\Rightarrow\quad \theta=144°, 216°, \ldots$$

$$\cos\theta=\frac{(-1+\sqrt{5})}{4} \quad\Rightarrow\quad \theta=72°, 288° \quad\Rightarrow\quad \mathbf{\cos 72=\frac{\sqrt{5}-1}{4}} \quad and \quad \boldsymbol{p=4}$$

6 The line l_1 has equation $\mathbf{r}=\begin{bmatrix}4\\-5\\3\end{bmatrix}+\lambda\begin{bmatrix}-1\\3\\1\end{bmatrix}$.

The line l_2 has equation $\mathbf{r}=\begin{bmatrix}7\\-8\\6\end{bmatrix}+\mu\begin{bmatrix}2\\-3\\1\end{bmatrix}$.

The point P lies on l_1 where $\lambda=-1$. The point Q lies on l_2 where $\mu=2$.

(a) Show that the vector $\overrightarrow{PQ}$ is parallel to $\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

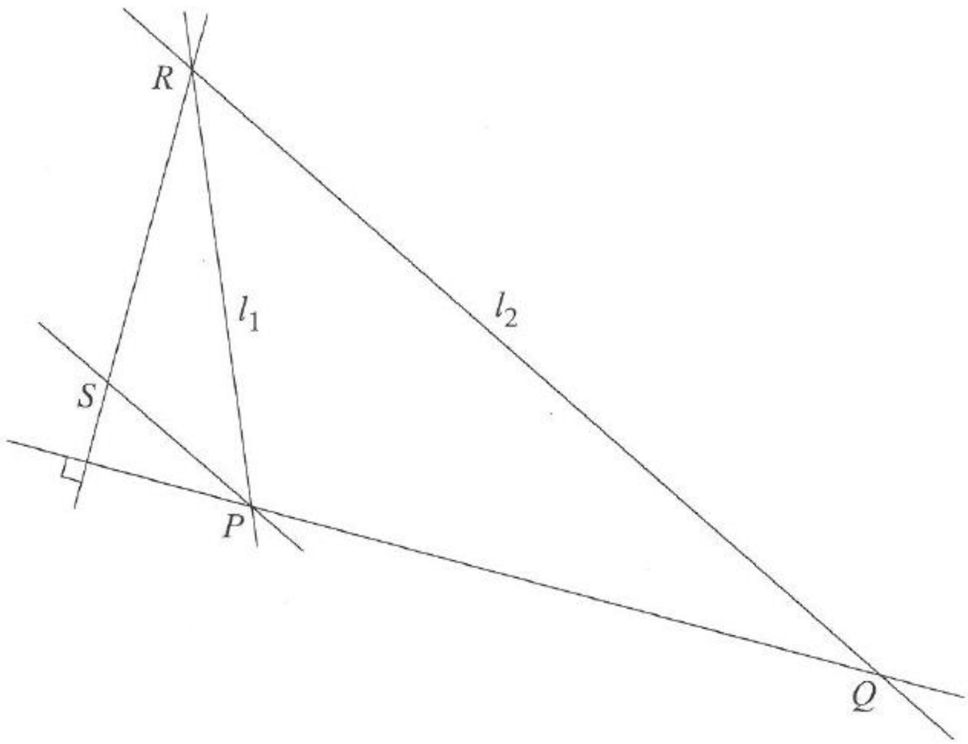
[3 marks]

(b) The lines l_1 and l_2 intersect at the point $R(3, b, c)$.

(i) Show that $b=-2$ and find the value of c.

[3 marks]

(ii) The point S lies on a line through P that is parallel to l_2. The line RS is perpendicular to the line PQ.



Find the coordinates of S.

[4 marks]

6.

a)

$$\overrightarrow{OP} = \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 2 \end{bmatrix} \qquad \overrightarrow{OQ} = \begin{bmatrix} 7 \\ -8 \\ 6 \end{bmatrix} + (2)\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -14 \\ 8 \end{bmatrix}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 11 \\ -14 \\ 8 \end{bmatrix} - \begin{bmatrix} 5 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} = \mathbf{6}\begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{\overrightarrow{PQ}\ parallel\ to} \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}$$

b)

i.

$$l_1 \ and \ l_2 \ intersect\ when: \quad \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix} + \lambda\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 6 \end{bmatrix} + \mu\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4-\lambda \\ -5+3\lambda \\ 3+\lambda \end{bmatrix} = \begin{bmatrix} 7+2\mu \\ -8-3\mu \\ 6+\mu \end{bmatrix}$$

$$3+\lambda = 6+\mu \quad \Rightarrow \quad \lambda = 3+\mu$$

$$4-\lambda = 7+2\mu \quad \Rightarrow \quad 4-(3+\mu) = 7+2\mu \quad \Rightarrow \quad \mu = -2 \quad \Rightarrow \quad \lambda = 1$$

$$-5+3(1) = -2 \ and \ -8-3(-2) = -2 \Rightarrow Lines\ intersect\ when\ \lambda = 1\ (or, equivalently, when\ \mu = -2)$$

$$\lambda = 1 \quad \Rightarrow \quad \begin{bmatrix} 4-\lambda \\ -5+3\lambda \\ 3+\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \\ \mathbf{4} \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} \quad \boldsymbol{where\ b = -2\ \ and\ \ c = 4}$$

ii.

$$\overrightarrow{OS} = \begin{bmatrix} 5 \\ -8 \\ 2 \end{bmatrix} + k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5+2k \\ -8-3k \\ 2+k \end{bmatrix} \quad and \quad \overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 5+2k \\ -8-3k \\ 2+k \end{bmatrix} = \begin{bmatrix} -2-2k \\ 6+3k \\ 2-k \end{bmatrix}$$

$$\begin{bmatrix} -2-2k \\ 6+3k \\ 2-k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad k = -1 \quad \Rightarrow \quad \overrightarrow{OS} = \begin{bmatrix} 5+2(-1) \\ -8-3(-1) \\ 2+(-1) \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{S}: (\mathbf{3}, \mathbf{-5}, \mathbf{1})$$

7 A curve has equation $\cos 2y + ye^{3x} = 2\pi$.

The point $A\left(\ln 2, \frac{\pi}{4}\right)$ lies on this curve.

(a) (i) Find an expression for $\frac{dy}{dx}$.

[6 marks]

(ii) Hence find the exact value of the gradient of the curve at A.

[1 mark]

(b) The normal at A crosses the y-axis at the point B. Find the exact value of the y-coordinate of B.

[2 marks]

7.

a)

i.

$$\cos 2y + ye^{3x} = 2\pi \quad \Rightarrow \quad -2\sin 2y\frac{dy}{dx} + y(3e^{3x}) + e^{3x}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx}(e^{3x} - 2\sin 2y) = -3ye^{3x}$$

$$\Rightarrow \quad \boldsymbol{\frac{dy}{dx} = \frac{-3ye^{3x}}{e^{3x} - 2\sin 2y}}$$

ii.

$$x = \ln 2 \ \ and \ \ y = \frac{\pi}{4} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-3\left(\frac{\pi}{4}\right)e^{3(\ln 2)}}{e^{3(\ln 2)} - 2\sin 2\left(\frac{\pi}{4}\right)} = \frac{-\frac{3\pi}{4}e^{\ln 8}}{e^{\ln 8} - 2\sin\frac{\pi}{2}} = \frac{-\frac{3\pi}{4}(8)}{8-2} = \frac{-6\pi}{6} = \boldsymbol{-\pi}$$

b)

$$\frac{dy}{dx} = -\pi \quad \Rightarrow \quad Gradient\ of\ normal = \frac{1}{\pi}$$

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - \frac{\pi}{4} = \frac{1}{\pi}(x - \ln 2)$$

$$Crosses\ y-axis\ when\ x = 0 \quad \Rightarrow \quad y - \frac{\pi}{4} = \frac{1}{\pi}(0 - \ln 2) \quad \Rightarrow \quad y - \frac{\pi}{4} = \frac{-\ln 2}{\pi} \quad \Rightarrow \quad \boldsymbol{y = \frac{\pi}{4} - \frac{\ln 2}{\pi}}$$

8 (a) Express $\dfrac{16x}{(1-3x)(1+x)^2}$ in the form $\dfrac{A}{1-3x}+\dfrac{B}{1+x}+\dfrac{C}{(1+x)^2}$.

[4 marks]

(b) Solve the differential equation

$$\frac{dy}{dx}=\frac{16x e^{2y}}{(1-3x)(1+x)^2}$$

where $y=0$ when $x=0$.

Give your answer in the form $f(y)=g(x)$.

[7 marks]

8.

a)

$$\frac{16x}{(1-3x)(1+x)^2}=\frac{A}{1-3x}+\frac{B}{1+x}+\frac{C}{(1+x)^2} \quad\Rightarrow\quad 16x=A(1+x)^2+B(1+x)(1-3x)+C(1-3x)$$

$$At\ x=-1:\ -16=4C \quad\Rightarrow\quad C=-4 \qquad At\ x=\frac{1}{3}:\ \frac{16}{3}=\frac{16A}{9} \quad\Rightarrow\quad A=3$$

$$Comparing\ coefficients:\ \ 0=A+B+C \quad\Rightarrow\quad B=1$$

$$\Rightarrow\quad \frac{16x}{(1-3x)(1+x)^2}=\mathbf{\frac{3}{1-3x}+\frac{1}{1+x}+\frac{-4}{(1+x)^2}}$$

b)

$$\frac{dy}{dx}=\frac{16xe^{2y}}{(1-3x)(1+x)^2} \quad\Rightarrow\quad \int e^{-2y}\,dy=\int\frac{16x}{(1-3x)(1+x)^2}\,dx$$

$$\Rightarrow\quad -\frac{e^{-2y}}{2}=\int\frac{3}{1-3x}+\frac{1}{1+x}+\frac{-4}{(1+x)^2}\,dx=-\int\frac{-3}{1-3x}\,dx+\int\frac{1}{1+x}\,dx-4\int(1+x)^{-2}\,dx$$

$$-\frac{e^{-2y}}{2}=-\ln(1-3x)+\ln(1+x)-4(-(1+x)^{-1})+C$$

$$y=0\ \ when\ \ x=0 \quad\Rightarrow\quad -\frac{1}{2}=0+0-4(-(1)^{-1})+C \quad\Rightarrow\quad -\frac{9}{2}=C$$

$$\Rightarrow\quad \boldsymbol{e^{-2y}=2\ln\frac{1-3x}{1+x}-\frac{8}{1+x}+9}$$