AQA MFP3 Further Pure 3 Further Mathematics 19 May 2014

Question Paper and Worked Solutions

Please note, this document represents my own solutions to the questions, is entirely unofficial and is not related to the mark scheme (which I have not seen). Therefore, while it should help you see how to do the questions, it won't include every valid method or give you a break down of the mark allocation. If you spot any errors, or think you have found a better solution, please [email me] so I can update it.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \frac{\ln(x+y)}{\ln y}$$

and $$y(6) = 3$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.4$, to obtain an approximation to $y(6.4)$, giving your answer to three decimal places.

[5 marks]

1.

$$\frac{dy}{dx} = f(x,y) = \frac{\ln(x+y)}{\ln y} \qquad x_0 = 6, y_0 = 3 \qquad h = 0.4 \quad \Rightarrow \quad x_1 = 3.4, y_1 = y(3.4)$$

Substituting into formulae:

$$k_1 = 0.4f(6,3) = \frac{0.4\ln 9}{\ln 3} = \frac{0.8\ln 3}{\ln 3} = 0.8$$

$$k_2 = 0.4f(x_0 + 0.4, y_0 + 0.8) = \frac{0.4\ln(6.4 + 3 + 0.8)}{\ln 3.8} = \frac{0.4\ln 10.2}{\ln 3.8}$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 3 + \frac{1}{2}\left(0.8 + \frac{0.4\ln 10.2}{\ln 3.8}\right) = \mathbf{3.748\ to\ 3\ d.p.}$$

2 (a) Find the values of the constants a, b and c for which $a + b\sin 2x + c\cos 2x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 20 - 20\cos 2x$$

[4 marks]

(b) Hence find the solution of this differential equation, given that $y = 4$ when $x = 0$.

[4 marks]

2.

a)

$$y = a + b\sin 2x + c\cos 2x \quad \Rightarrow \quad \frac{dy}{dx} = 2b\cos 2x - 2c\sin 2x$$

$$\Rightarrow \quad \frac{dy}{dx} + 4y = 2b\cos 2x - 2c\sin 2x + 4(a + b\sin 2x + c\cos 2x) = 20 - 20\cos 2x$$

$$\Rightarrow \quad 4a + (4b - 2c)\sin 2x + (4c + 2b)\cos 2x = 20 - 20\cos 2x$$

$$\Rightarrow \quad 4a = 20 \quad \Rightarrow \quad \boldsymbol{a = 5}$$

$$4b - 2c = 0 \quad \Rightarrow \quad c = 2b$$

$$4c + 2b = -20 \quad \Rightarrow \quad 2c + b = -10 \quad \Rightarrow \quad 2(2b) + b = -10 \quad \Rightarrow \quad \boldsymbol{b = -2} \quad \Rightarrow \quad \boldsymbol{c = -4}$$

b)

Finding the complementary function:

$$\frac{dy}{dx} + 4y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -4y \quad \Rightarrow \quad \int \frac{1}{y}\,dy = \int -4\,dx \quad \Rightarrow \quad \ln y = -4x + C \quad \Rightarrow \quad y = e^{-4x+C} = Ke^{-4x}$$

General solution:

$$y = Ke^{-4x} + 5 - 2\sin 2x - 4\cos 2x$$

$$y = 4, x = 0 \quad \Rightarrow \quad 4 = Ke^{-4(0)} + 5 - 2\sin 2(0) - 4\cos 2(0)$$

$$4 = K + 5 - 4 \quad \Rightarrow \quad K = 3$$

$$\boldsymbol{y = 3e^{-4x} + 5 - 2\sin 2x - 4\cos 2x}$$

3 A curve has polar equation $r(4 - 3\cos\theta) = 4$. Find its Cartesian equation in the form $y^2 = \mathrm{f}(x)$.

[4 marks]

3.

$$r(4 - 3\cos\theta) = 4 \quad \Rightarrow \quad 4r - 3r\cos\theta = 4 \quad \Rightarrow \quad 4r = 4 + 3r\cos\theta \quad \Rightarrow \quad 16r^2 = (4 + 3r\cos\theta)^2$$

$$\Rightarrow \quad 16(x^2 + y^2) = (4 + 3x)^2 \quad \Rightarrow \quad 16x^2 + 16y^2 = (4 + 3x)^2 \quad \Rightarrow \quad \boldsymbol{y^2 = \frac{(4 + 3x)^2}{16} - x^2}$$

4 Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{-x}$$

given that $y \to 0$ as $x \to \infty$ and that $\frac{dy}{dx} = -3$ when $x = 0$.

[10 marks]

4.

Characteristic equation to find complementary function:

$$k^2 - 2k - 3 = 0$$

$$(k-3)(k+1) = 0 \quad \Rightarrow \quad k = 3 \quad or \quad k = -1$$

$$\Rightarrow \quad CF: \quad y_c = Ae^{3x} + Be^{-x}$$

Trial function:

$$y = axe^{\lambda x} \quad \Rightarrow \quad \frac{dy}{dx} = a\left(\lambda xe^{\lambda x} + e^{\lambda x}\right) \quad \Rightarrow \quad \frac{d^2y}{dx^2} = a\lambda\left(\lambda xe^{\lambda x} + e^{\lambda x}\right) + a\lambda e^{\lambda x} = a\lambda e^{\lambda x}(\lambda x + 2)$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = a\lambda e^{\lambda x}(\lambda x + 2) - 2a\left(\lambda xe^{\lambda x} + e^{\lambda x}\right) - 3axe^{\lambda x}$$

$$= e^{\lambda x}(a\lambda^2 x + 2a\lambda - 2a\lambda x - 2a - 3ax) = ae^{\lambda x}\left((\lambda^2 - 2\lambda - 3)x + 2(\lambda - 1)\right) = 2e^{-x}$$

$$\Rightarrow \quad \lambda = -1 \quad \Rightarrow \quad -4a = 2 \quad \Rightarrow \quad a = -\frac{1}{2}$$

$$\Rightarrow \quad PI: \quad y_p = -\frac{1}{2}xe^{-x}$$

General solution:

$$GS = PI + CF: \quad y = Ae^{3x} + Be^{-x} - \frac{1}{2}xe^{-x}$$

$$\frac{dy}{dx} = -3 \ when \ x = 0 \quad \Rightarrow \quad 3Ae^{3(0)} - Be^{-(0)} - \frac{1}{2}\left(-(0)e^{-(0)} + e^{-(0)}\right) = -3$$

$$3A - B - \frac{1}{2}(1) = -3$$

Also note that since $y \to 0$ as $x \to \infty$, $A = 0$ (otherwise the e^{3x} term would $\to \infty$).

$$\Rightarrow \quad B + \frac{1}{2} = 3 \quad \Rightarrow \quad B = \frac{5}{2}$$

$$\boldsymbol{y = \frac{5}{2}e^{-x} - \frac{1}{2}xe^{-x}}$$

5 (a) Find $\int x \cos 8x \, dx$. **[3 marks]**

(b) Find $\lim_{x \to 0} \left[\frac{1}{x} \sin 2x\right]$. **[2 marks]**

(c) Explain why $\int_0^{\frac{\pi}{4}} \left(2 \cot 2x - \frac{1}{x} + x \cos 8x\right) dx$ is an improper integral. **[1 mark]**

(d) Evaluate $\int_0^{\frac{\pi}{4}} \left(2 \cot 2x - \frac{1}{x} + x \cos 8x\right) dx$, showing the limiting process used. Give your answer as a single term. **[4 marks]**

5.

a)

$$\int x \cos 8x \;\, dx$$

Using integration by parts:

$$u = x \quad \frac{dv}{dx} = \cos 8x$$

$$\frac{du}{dx} = 1 \quad v = \frac{\sin 8x}{8}$$

$$\int x \cos 8x \;\, dx = \frac{x \sin 8x}{8} - \int \frac{\sin 8x}{8} \;\, dx = \boldsymbol{\frac{x \sin 8x}{8} + \frac{\cos 8x}{64} + C}$$

b)

Using Maclaurin Series:

$$\lim_{x \to 0} \left[\frac{1}{x} \sin 2x\right] = \lim_{x \to 0} \left[\frac{1}{x}\left((2x) - \frac{(2x)^3}{3!} + O(x^5)\right)\right] = \lim_{x \to 0} \left[2 - \frac{8x^2}{3!} + O(x^4)\right] = \mathbf{2}$$

c)

$\int_0^{\frac{\pi}{4}} 2 \cot 2x - \frac{1}{x} + x \cos 8x \;\, dx$ is an improper integral because neither $\cot 2x$ nor $\frac{1}{x}$ is defined at the lower limit, 0.

d)

$$\int_0^{\frac{\pi}{4}} 2 \cot 2x - \frac{1}{x} + x \cos 8x \;\, dx = \lim_{a \to 0} \left[\int_a^{\frac{\pi}{4}} 2 \cot 2x - \frac{1}{x} + x \cos 8x \;\, dx\right]$$

$$= \lim_{a \to 0} \left[\left[\ln(\sin 2x) - \ln x + \frac{x \sin 8x}{8} + \frac{\cos 8x}{64}\right]_a^{\frac{\pi}{4}}\right] = \lim_{a \to 0} \left[\left[\ln\left(\frac{\sin 2x}{x}\right) + \frac{1}{64}(8x \sin 8x + \cos 8x)\right]_a^{\frac{\pi}{4}}\right]$$

$$= \lim_{a \to 0} \left[\left(\ln\left(\frac{\sin \frac{\pi}{2}}{\frac{\pi}{4}}\right) + \frac{1}{64}(2\pi \sin 2\pi + \cos 2\pi)\right) - \left(\ln\left(\frac{\sin 2a}{a}\right) + \frac{1}{64}(8a \sin 8a + \cos 8a)\right)\right]$$

$$= \lim_{a \to 0} \left[\left(\ln\left(\frac{4}{\pi}\right) + \frac{1}{64}\right) - \left(\ln\left(\frac{\sin 2a}{a}\right) + \frac{1}{64}(8a \sin 8a + \cos 8a)\right)\right]$$

$$= \ln\left(\frac{4}{\pi}\right) + \frac{1}{64} - \lim_{a \to 0} \left[\ln\left(\frac{\sin 2a}{a}\right)\right] - \lim_{a \to 0} \left[\frac{(8a \sin 8a + \cos 8a)}{64}\right] = \ln\left(\frac{4}{\pi}\right) + \frac{1}{64} - \ln 2 - \frac{1}{64} = \boldsymbol{\ln \frac{2}{\pi}}$$

6 (a) By using an integrating factor, find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4}u = 3(x^2+4)$$

giving your answer in the form $u = \mathrm{f}(x)$.

[6 marks]

(b) Show that the substitution $u = x^2\frac{\mathrm{d}y}{\mathrm{d}x}$ transforms the differential equation

$$x^2(x^2+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x^2+4)^2$$

into

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4}u = 3(x^2+4)$$

[4 marks]

(c) Hence, given that $x > 0$, find the general solution of the differential equation

$$x^2(x^2+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x^2+4)^2$$

[2 marks]

6.

a)

$$IF = e^{\int \frac{-2x}{x^2+4}dx} = e^{-\ln(x^2+4)} = \frac{1}{x^2+4}$$

$$\frac{1}{x^2+4}\frac{du}{dx} - \frac{2x}{(x^2+4)^2}u = 3$$

$$\frac{1}{x^2+4}u = \int 3\,dx \quad \Rightarrow \quad \frac{1}{x^2+4}u = 3x + C \quad \Rightarrow \quad \boldsymbol{u = (x^2+4)(3x+C)}$$

b)

$$u = x^2\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = ux^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -2x^{-3}u + x^{-2}\frac{du}{dx}$$

$$x^2(x^2+4)\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} = 3(x^2+4)^2 \quad \Rightarrow \quad x^2(x^2+4)\left(-2x^{-3}u + x^{-2}\frac{du}{dx}\right) + 8x(ux^{-2}) = 3(x^2+4)^2$$

$$\Rightarrow \quad (x^2+4)\left(-2x^{-1}u + \frac{du}{dx}\right) + \frac{8u}{x} = 3(x^2+4)^2 \quad \Rightarrow \quad -\frac{2u(x^2+4)}{x} + (x^2+4)\frac{du}{dx} + \frac{8u}{x} = 3(x^2+4)^2$$

$$\Rightarrow \quad -\frac{2u}{x} + \frac{du}{dx} + \frac{8u}{x(x^2+4)} = 3(x^2+4) \quad \Rightarrow \quad \frac{du}{dx} + \frac{(8u - 2u(x^2+4))}{x(x^2+4)} = 3(x^2+4)$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{8u - 2ux^2 - 8u}{x(x^2+4)} = 3(x^2+4) \quad \Rightarrow \quad \boldsymbol{\frac{du}{dx} - \frac{2x}{x^2+4}u = 3(x^2+4)}$$

c)

$$u = (x^2+4)(3x+C) \quad \Rightarrow \quad x^2\frac{dy}{dx} = (x^2+4)(3x+C) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(x^2+4)(3x+C)}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x + 12x^{-1} + C + 4Cx^{-2} \quad \Rightarrow \quad \boldsymbol{y = \frac{3x^2}{2} + 12\ln x + Cx - 4Cx^{-1} + D}$$

7 (a) It is given that $y = \ln(\cos x + \sin x)$.

(i) Show that $\dfrac{d^2y}{dx^2} = -\dfrac{2}{1+\sin 2x}$.

[4 marks]

(ii) Find $\dfrac{d^3y}{dx^3}$.

[1 mark]

(b) (i) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x + \sin x)$ are $x - x^2 + \frac{2}{3}x^3$.

[3 marks]

(ii) Write down the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x - \sin x)$.

[1 mark]

(c) **Hence** find the first three non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\dfrac{\cos 2x}{e^{3x-1}}\right)$.

[4 marks]

7.

a)

i.

$$y = \ln(\cos x + \sin x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\sin x + \cos x}{\cos x + \sin x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(\cos x + \sin x)(-\cos x - \sin x) - (-\sin x + \cos x)(-\sin x + \cos x)}{(\cos x + \sin x)^2}$$

$$= \frac{-\cos^2 x - 2\sin x\cos x - \sin^2 x - (\sin^2 x - 2\sin x\cos x + \cos^2 x)}{(\cos x + \sin x)^2} = \frac{-2}{(\cos x + \sin x)^2}$$

$$= -\frac{2}{\cos^2 x + 2\sin x\cos x + \sin^2 x} = -\mathbf{\frac{2}{1+\sin 2x}}$$

ii.

$$\frac{d^3y}{dx^3} = \frac{\big((1+\sin 2x)(0) - (-2)(2\cos 2x)\big)}{(1+\sin 2x)^2} = \mathbf{\frac{4\cos 2x}{(1+\sin 2x)^2}}$$

b)

i.

$$f(0) = \ln(1+0) = 0 \qquad f'(0) = \frac{-0+1}{1+0} = 1 \qquad f''(0) = \frac{-2}{(1+0)^2} = -2 \qquad f'''(0) = \frac{4(1)}{(1+0)^2} = 4$$

$$\ln(\cos x + \sin x) \approx 0 + 1x + \frac{(-2)x^2}{2!} + \frac{4x^3}{3!} = \mathbf{x - x^2 + \frac{2x^3}{3}}$$

ii.

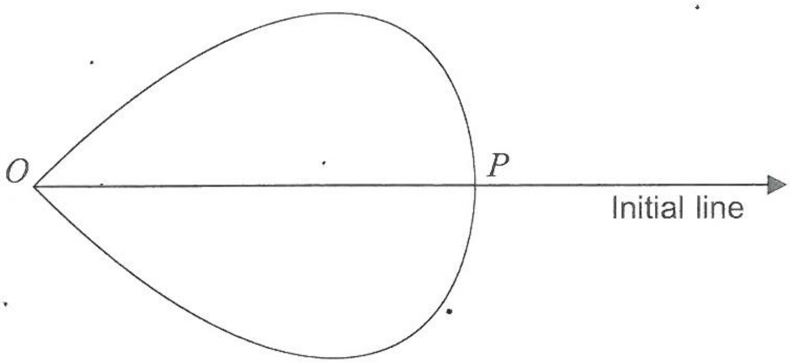
$$\ln(\cos x - \sin x) = \ln(\cos(-x) + \sin(-x)) \approx (-x) - (-x)^2 + \frac{2(-x)^3}{3} = \mathbf{-x - x^2 - \frac{2x^3}{3}}$$

c)

$$\ln\left(\frac{\cos 2x}{e^{3x-1}}\right) = \ln(\cos^2 x - \sin^2 x) - (3x-1) = \ln(\cos x + \sin x)(\cos x - \sin x) - 3x + 1$$

$$= \ln(\cos x + \sin x) + \ln(\cos x - \sin x) - 3x + 1 \approx x - x^2 + \frac{2x^3}{3} + \left(-x - x^2 - \frac{2x^3}{3}\right) - 3x + 1 = \mathbf{1 - 3x - 2x^2}$$

8 The diagram shows a sketch of a curve C, the pole O and the initial line. The curve C intersects the initial line at the point P.



The polar equation of C is $r = (1 - \tan^2\theta)\sec\theta$, $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$.

(a) Show that the area of the region bounded by the curve C is $\frac{8}{15}$.

[5 marks]

(b) The curve whose polar equation is

$$r = \frac{1}{2}\sec^3\theta, \quad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

intersects C at the points A and B.

(i) Find the polar coordinates of A and B.

[3 marks]

(ii) Given that angle OAP = angle $OBP = \alpha$, show that $\tan\alpha = k\sqrt{3}$, where k is an integer.

[4 marks]

(iii) Using your value of k from part **(b)(ii)**, state whether the point A lies inside or lies outside the circle whose diameter is OP. Give a reason for your answer.

[1 mark]

8.
a)

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}r^2\, d\theta = \int_0^{\frac{\pi}{4}} (1 - \tan^2\theta)^2 \sec^2\theta \; d\theta$$

Using substitution:

$$u = \tan\theta \quad \Rightarrow \quad \frac{du}{d\theta} = \sec^2\theta$$

$$A = \int_{\tan 0}^{\tan\frac{\pi}{4}} (1 - u^2)^2\, du = \int_0^1 1 - 2u^2 + u^4\, du = \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \mathbf{\frac{8}{15}}$$

b)
i.

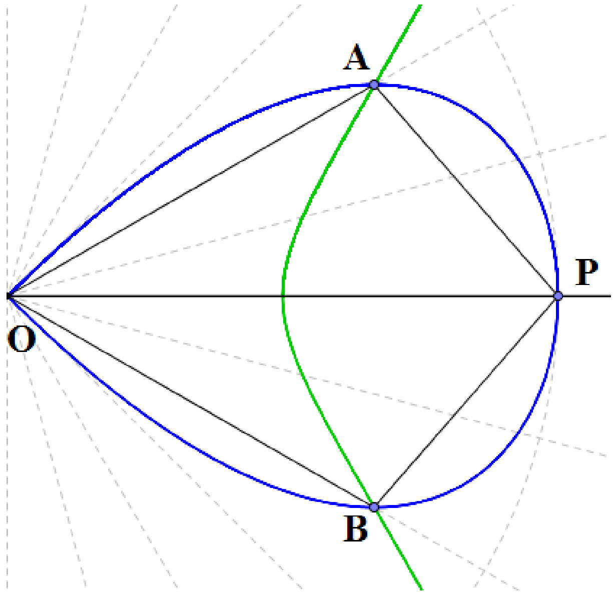
$$\frac{1}{2}\sec^3\theta = (1 - \tan^2\theta)\sec\theta \quad \Rightarrow \frac{1}{2}\sec^3\theta - \sec\theta + \sec\theta\tan^2\theta = 0 \quad \Rightarrow \quad \sec\theta\left(\frac{1}{2}\sec^2\theta - 1 + \tan^2\theta\right) = 0$$

$$\Rightarrow \quad \sec\theta = 0 \quad (no\ solutions) \quad or \quad \frac{1}{2}\sec^2\theta - 1 + \tan^2\theta = 0 \quad \Rightarrow \sec^2\theta - 2 + 2\tan^2\theta = 0$$

$$\Rightarrow \quad \sec^2\theta - 2 + 2(\sec^2\theta - 1) = 0 \quad \Rightarrow \quad 3\sec^2\theta = 4 \quad \Rightarrow \quad \sec\theta = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{6}$$

$$r = \frac{1}{2}\sec^3\left(\pm\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9} \quad \Rightarrow \quad Crossing\ points: \ \left(\boldsymbol{\frac{4\sqrt{3}}{9}, \frac{\pi}{6}}\right) \ \boldsymbol{and} \ \left(\boldsymbol{\frac{4\sqrt{3}}{9}, -\frac{\pi}{6}}\right)$$

ii.



$$P\hat{O}A = \frac{\pi}{6} \qquad OA = \frac{4\sqrt{3}}{9} \qquad OP = (1 - \tan^2 0)\sec 0 = 1$$

Using cosine rule:

$$AP^2 = \left(\frac{4\sqrt{3}}{9}\right)^2 + 1^2 - 2\left(\frac{4\sqrt{3}}{9}\right)(1)\cos\frac{\pi}{6}$$

$$AP = \sqrt{\frac{16}{27} + 1 - \frac{8\sqrt{3}}{9}\left(\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{7}{27}} = \frac{\sqrt{7}}{3\sqrt{3}} = \frac{\sqrt{21}}{9}$$

Using sine rule:

$$\frac{1}{\sin\alpha} = \frac{\frac{\sqrt{21}}{9}}{\sin\frac{\pi}{6}} \quad \Rightarrow \quad \sin\alpha = \frac{\frac{1}{2}}{\frac{\sqrt{21}}{9}} = \frac{9}{2\sqrt{21}} = \frac{9\sqrt{21}}{42}$$

Using cosine rule:

$$1^2 = \left(\frac{4\sqrt{3}}{9}\right)^2 + \left(\frac{\sqrt{21}}{9}\right)^2 - 2\left(\frac{4\sqrt{3}}{9}\right)\left(\frac{\sqrt{21}}{9}\right)\cos\alpha \quad \Rightarrow \quad 1 = \frac{16}{27} + \frac{7}{27} - \frac{24\sqrt{7}}{81}\cos\alpha$$

$$\Rightarrow \quad 81 = 69 - 24\sqrt{7}\cos\alpha \quad \Rightarrow \quad \frac{69 - 81}{24\sqrt{7}} = \cos\alpha = -\frac{1}{2\sqrt{7}}$$

$$\Rightarrow \quad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{9\sqrt{21}}{42}}{-\frac{1}{2\sqrt{7}}} = \boldsymbol{-3\sqrt{3}}$$

iii.

Since $k < 0$, the angle must be obtuse ($\tan\theta > 0$ for $0 < \theta < \frac{\pi}{2}$), and since the angle formed in a semicircle by end points of the diameter and the circumference is a right angle, the point A must lie inside the circle. The closer such a point is to the centre, the greater the angle becomes, and the further away, the smaller it becomes.