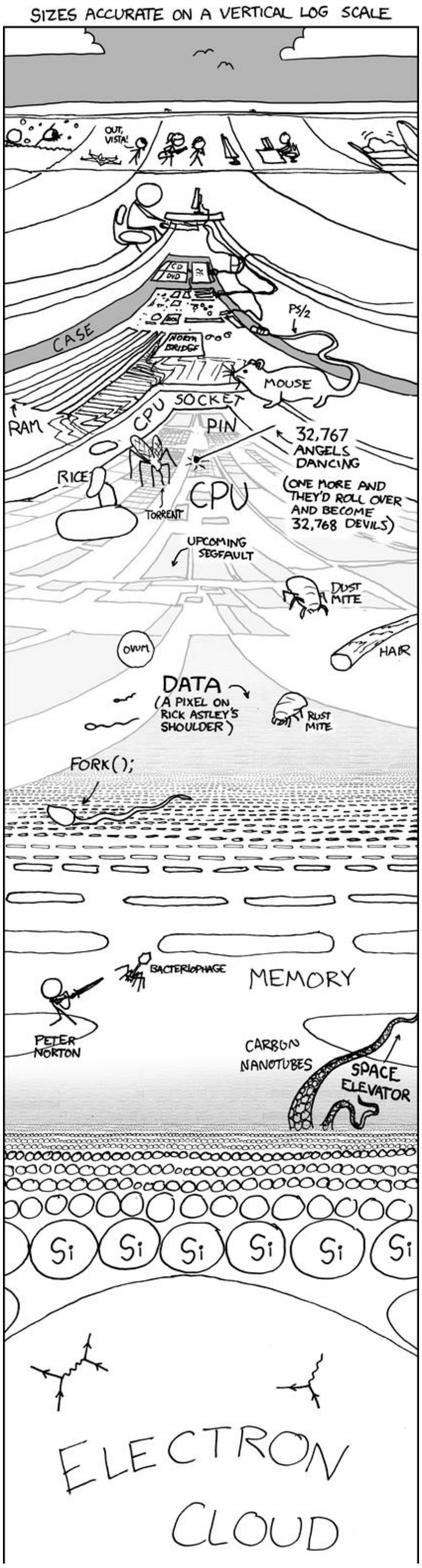
<u>Working with Logarithms</u>

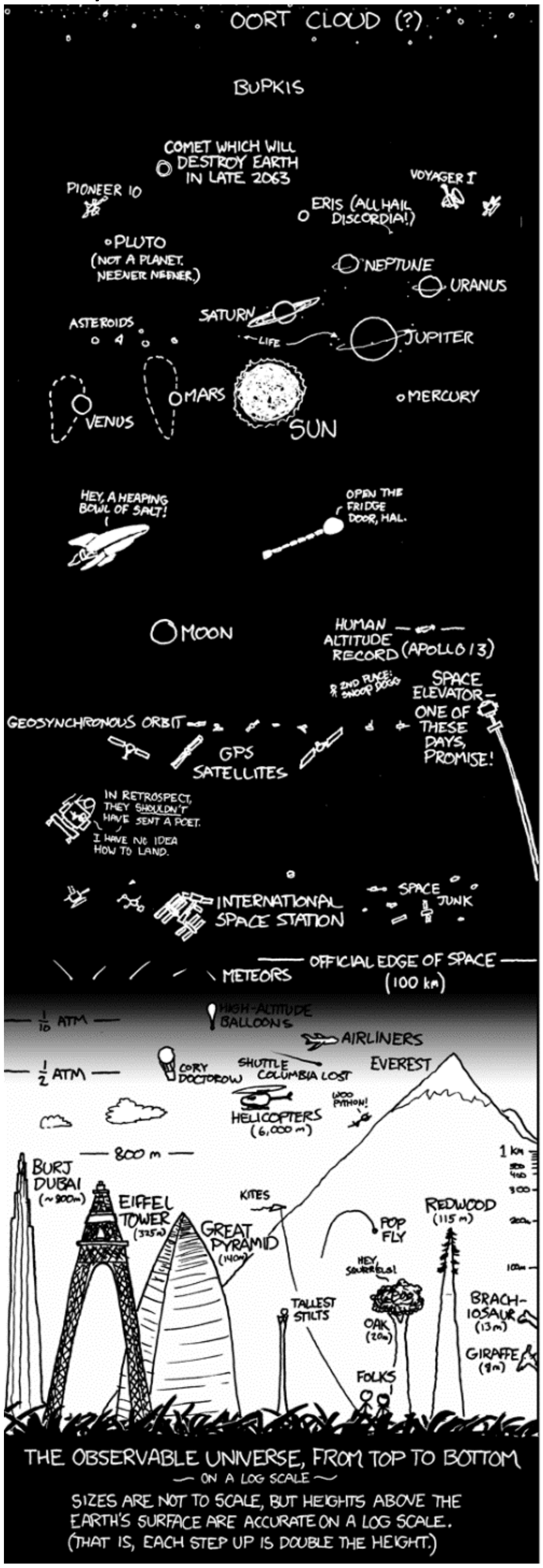
A Visual Log Scale

The multiplicative number line is often called a ‘logarithmic scale’, or ‘log scale’ for short. Many physical measurements are much easier to represent on a log scale than a linear one.

If we scaled the universe down in the normal way, a ping-pong ball sized Earth would require a 4 metre sphere for the sun, 500 metres away. Pluto would be the size of a marble, 20km further down the road. Tau Ceti (the nearest star with a possible Earth-like planet), would be a 3 metre sphere… on the moon!

To represent the relative sizes and distances of widely varying objects such as cosmic bodies or, indeed, microscopic objects, we need a consistent way to condense the number line:





(cropped: for the complete pictures, visit xkcd.com)

Using Logarithms to Solve Problems

The following problems all relate to compound interest.

The rule for calculating compound interest is:

$$Final\ Value = P \times I^n$$

where P is the principal, I the interest rate (eg 1.02 for 2%) and n the number of years.

Attempt the four questions using any method you like.

	Principal (original amount)	Interest rate (per year)	Years	Final amount
1.	£1000	2%	5	$\boldsymbol{a} = ?$
	If I invest £1000 at 2% interest for 5 years, how much will I have?			$a =$
2.	$\boldsymbol{b} = ?$	3.5%	6	£2950.21
	How much must I invest to get £2950.21 after 6 years at 3.5% interest?			$b =$
3.	£2000	$\boldsymbol{c} = ?$	10	£3000
	What interest rate would I need to grow £2000 into £3000 in 10 years?			$c =$
4.	£1200	4%	$\boldsymbol{d} = ?$	£2400
	How many years will it take to grow £1200 into £2400 at 4% interest?			$d =$

You will probably find that one of the values above required you to use trial and error. There is a better way, but it requires the use of a function that reverses exponentiation.

Three types of equation involving powers

The first two are straightforward; the last requires logarithms for all but the simplest cases.

Equation:	$4^3 = x$	$y^3 = 64$	$4^z = 64$
How to solve:	*Direct calculation* $x = 4 \times 4 \times 4 = \mathbf{64}$	*Inverse powers (roots)* $y = 64^{\frac{1}{3}} = \sqrt[3]{64} = \mathbf{4}$	*Logarithms* $z = log_4 64 = \mathbf{3}$

Commonly Used Logarithms

Some numbers are easy to take the logarithm of: we know that $\log_2 16 = 4$ because we recall that $2^4 = 16$, for instance. But for more awkward numbers, it takes a calculator.

Most calculators are set up to easily calculate logs with base 10 or base e, since these are the most commonly used bases. With many log problems you can choose the base, and even if you can't, there are ways to convert between bases.

Use the **log** button for base 10 logarithms. Check that **log 100** gives you **2**.

Use the **ln** button for the so-called natural log (base e). Check that **ln 100** gives about **4.6**.

Popular bases for logarithms:

Base:	**Used by:**	**Useful because:**	$\log 1234 =$
10	Physicists	Indicates the number of digits (eg the power when in standard form)	$3.091\ldots$
2	Programmers	Indicates the number of bits (eg the number of binary digits needed)	$10.269\ldots$
e $2.71\ldots$	Mathematicians	$y = e^x$ has equal gradient and y value (ie the rate of change is the height)	$7.118\ldots$

Base 10 tells us that 1234 has just over 3 digits, so it's 4 digits long.
Base 2 tells us we need just over 10 bits in binary, so 11 bits: 10011010010.
Base e will come in handy when we need to differentiate exponential functions, because $y = e^x$ has gradient $\frac{dy}{dx} = e^x$. The curve $y = e^x$ has slope 1234 at the point $(7.118, 1234)$.

How long is the longest known prime? $p = 300376418084\ldots391086436351$

The largest known prime, as of February 2016, is the massive number $\mathbf{2^{74,207,281} - 1}$. Your calculator struggles to deal with powers greater than 2 digits, but using logarithms we can find the approximate number of digits of this number in normal decimal notation.

Step 1: Ignore the -1. Powers of 2 never end in 0, so this won't affect our answer.

$$2^{74,207,281}$$

Step 2: Take logs, base 10. The answer, when rounded up, will give us the number of digits.

$$\log_{10} 2^{74,207,281}$$

Step 3: Use the log rule $n \log A = \log A^n$ to bring the power down in front, and calculate.

$$74207281 \log_{10} 2 = 22{,}338{,}617.48\ldots$$

Step 4: Round up to the next whole number (the 'ceiling') to give the number of digits:

$$\lceil 22{,}338{,}617.48\ldots \rceil = \boldsymbol{22,338,618\ digits}$$

Using Logarithms to Solve Problems SOLUTIONS

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Attempt the four questions using any method you like.

	Principal (original amount)	Interest rate (per year)	Years	Final amount
1.	£1000	2%	5	$\boldsymbol{a} = ?$
	If I invest £1000 at 2% interest for 5 years, how much will I have? $\boldsymbol{a} = 1000 \times 1.02^5 \approx \textbf{£1104.08}$			
2.	$\boldsymbol{b} = ?$	3.5%	6	£2950.21
	How much must I invest to get £2950.21 after 6 years at 3.5% interest? $\boldsymbol{b} = 2950.21 \div 1.035^6 \approx \textbf{£2400.00}$			
3.	£2000	$\boldsymbol{c} = ?$	10	£3000
	What interest rate would I need to grow £2000 into £3000 in 10 years? $\boldsymbol{c} = 1 - \left(\frac{3000}{2000}\right)^{\frac{1}{10}} \approx 0.04138 = \mathbf{4.138\%}$			
4.	£1200	4%	$\boldsymbol{d} = ?$	£2400
	How many years will it take to grow £1200 into £2400 at 4% interest? $\boldsymbol{d} = \log_{1.04}\left(\frac{2400}{1200}\right) \approx \boldsymbol{17.67\ years\ (or\ 17\ years, 246\ days)}$			

You will probably find that one of the values above required you to use trial and error. There is a better way, but it requires the use of a function that reverses exponentiation.

Three types of equation involving powers

The first two are straightforward; the last requires logarithms for all but the simplest cases.

Equation:	$4^3 = x$	$y^3 = 64$	$4^z = 64$
How to solve:	*Direct calculation* $x = 4 \times 4 \times 4 = \mathbf{64}$	*Inverse powers (roots)* $y = 64^{\frac{1}{3}} = \sqrt[3]{64} = \mathbf{4}$	*Logarithms* $z = log_4 64 = \mathbf{3}$

Taking Logs to Solve Equations

Equations like $4^x = 64$ are easily solvable because we can easily spot that $64 = 4^3$.

Equations like $4^x = 70$ are not so straightforward. But like any equation, provided we can find a way to *reverse* the process that has been applied to the unknown, we can solve it:

$$\boldsymbol{4^x = 70}$$

Take logs (base 4) of both sides:

$$\boldsymbol{x = \log_4 70 \approx 3.06}$$

Note that in this case, all you're really doing is recognising the definition of the logarithm.

This method is fine if your calculator can deal with different bases, but many calculators only have the capacity to find *log* and *ln*, so what if we took logs using a different base?

$$\boldsymbol{4^x = 70}$$

Take logs (base 10) of both sides:

$$\boldsymbol{\log_{10} 4^x = \log_{10} 70}$$

Simplifying LHS using the log rule $n \log A = \log A^n$*:*

$$\boldsymbol{x \log_{10} 4 = \log_{10} 70}$$

Dividing through by $\log_{10} 4$*:*

$$\boldsymbol{x = \frac{\log_{10} 70}{\log_{10} 4} \approx 3.06}$$

Notice that here we don't rely on any particular base. This is generally more convenient, but also gives us a new insight into logarithmic scales: the different scales are related *multiplicatively*. $\log_2 50$ and $\log_{10} 50$ are not equal, but $\frac{\log_2 50}{\log_2 30} = \frac{\log_{10} 50}{\log_{10} 30}$

How many years will it take for the value of my house to triple at 8% a year?

Formulate the problem as an equation:

$$\boldsymbol{x \times 1.08^n = 3x}$$

Divide through by x *(this shows that we didn't need to know the price, only that it triples):*

$$\boldsymbol{1.08^n = 3}$$

Taking logs of both sides:

$$\boldsymbol{\log_{10} 1.08^n = \log_{10} 3}$$

Applying the rule $log\ A^n = n\ log\ A$*:*

$$\boldsymbol{n \log_{10} 1.08 = \log_{10} 3}$$

Dividing through by $log_{10}\ 1.08$*:*

$$\boldsymbol{n = \frac{\log_{10} 3}{\log_{10} 1.08} \approx 14.3\ years}$$

Logarithm Exam Questions

The following questions are all taken from AQA Core 2 exams.

Laws of Logs (not provided in the formula book)
$\log A + \log B = \log AB$
$\log A - \log B = \log \frac{A}{B}$
$\log A^n = n \log A$
$\log 10 = 1 \quad \& \quad \log 1 = 0$

Question	***Hint***
1. Use logarithms to solve the equation $0.8^x = 0.05$, giving your answer to three decimal places.	*Take logs of both sides and make use of the rule for powers in logs: $log\ A^n = n\ log\ A$.*
2. Given that $\log_a x = 3\log_a 6 - \log_a 8$ where a is a positive constant, find x.	*Use the rule $log\ A^n = n\ log\ A$ to simplify $3\ log_a 6$ first, then use the rule for adding logarithms: $log\ A + log\ B = log\ AB$.*
3. It is given that n solves the equation $2\log_a n - \log_a(5n - 24) = \log_a 4$. Show that $n^2 - 20n + 96 = 0$, and hence find n.	*Use the rule for powers in logs to simplify $2\ log_a n$, then the rule for subtracting logarithms: $log\ A - log\ B = log\frac{A}{B}$.*

Question	***Hint***
4. Given that $\log_a y + \log_a 5 = 7$, express y in terms of a, giving your answer in a form not involving logarithms.	*Combine the terms on the right into a single logarithm using the rule for adding logs. Next, reverse the function by using the opposite exponential function.*
5. Express xy in terms of a, given that $\log_a x = 3$ and $\log_a y - 3\log_a 2 = 4$.	*Write x as a power of a, then find y as a power of a by combining the left-hand-side first using log rules.*
6. Given that $\log_2 p = m$ and $\log_8 q = n$, express pq in the form 2^y where y is an expression in m and n.	*Use the inverse function 8^x to write q in terms of n, first as a power of 8, then – by expressing 8 as a power of 2, as a power of 2 itself. Find an expression for p in terms of m, and combine using index laws.*
7. The curves $y = 4^{-x}$ and $y = 3 \times 4^x$ intersect at exactly one point. Find the x coordinate of this point of intersection.	*Equate the right-hand-sides, and take logs. Use the log rule involving powers in logs to isolate x, then rearrange.*

Logarithm Exam Questions SOLUTIONS

The following questions are all taken from AQA Core 2 exams.

Question	*Solution*
1. Use logarithms to solve the equation $0.8^x = 0.05$, giving your answer to three decimal places.	$\ln 0.8^x = \ln 0.05$ $x \ln 0.8 = \ln 0.05$ $x = \dfrac{\ln 0.05}{\ln 0.8} \approx \mathbf{13.425}$
2. Given that $\log_a x = 3\log_a 6 - \log_a 8$ where a is a positive constant, find x.	$\log_a x = \log_a 6^3 - \log_a 8$ $\log_a x = \log_a \dfrac{6^3}{8}$ $x = \dfrac{6^3}{8} = \mathbf{27}$
3. It is given that n solves the equation $2\log_a n - \log_a(5n - 24) = \log_a 4$. Show that $n^2 - 20n + 96 = 0$, and hence find n.	$\log_a n^2 - \log_a(5n - 24) = \log_a 4$ $\log_a \dfrac{n^2}{5n-24} = \log_a 4$ $\dfrac{n^2}{5n-24} = 4$ $n^2 = 20n - 96$ $\boldsymbol{n^2 - 20n + 96 = 0\ as\ required}$
4. Given that $\log_a y + \log_a 5 = 7$, express y in terms of a, giving your answer in a form not involving logarithms.	$\log_a 5y = 7$ $5y = a^7$ $\boldsymbol{y = \dfrac{a^7}{5}}$
5. Express xy in terms of a, given that $\log_a x = 3$ and $\log_a y - 3\log_a 2 = 4$.	$x = a^3$ $\log_a y - \log_a 2^3 = 4$ $\log_a \dfrac{y}{2^3} = 4$ $\dfrac{y}{8} = a^4$ $y = 8a^4$ $xy = a^3 \times 8a^4 = \boldsymbol{8a^7}$
6. Given that $\log_2 p = m$ and $\log_8 q = n$, express pq in the form 2^y where y is an expression in m and n.	$p = 2^m$ $q = 8^n = (2^3)^n = 2^{3n}$ $pq = 2^m \times 2^{3n} = \boldsymbol{2^{m+3n}}$
7. The curves $y = 4^{-x}$ and $y = 3 \times 4^x$ intersect at exactly one point. Find the x coordinate of this point of intersection.	$4^{-x} = 3 \times 4^x$ $\ln 4^{-x} = \ln(3 \times 4^x)$ $-x\ln 4 = \ln 3 + x\ln 4$ $2x\ln 4 = -\ln 3$ $\boldsymbol{x = -\dfrac{\ln 3}{2\ln 4}}$