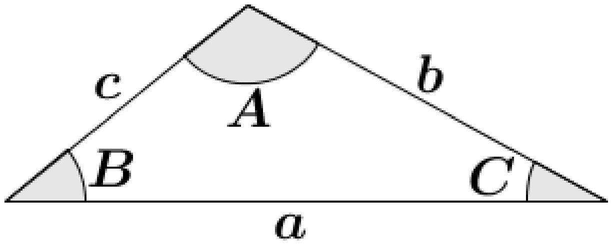
Understanding the Sine Rule

For a triangle labelled as shown, the following three ratios are equal:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Deriving the Sine Rule

The formula for area is usually given as $\frac{1}{2}ab\sin C$, but as long as the angle used is in between the two sides used, any of these three is equivalent, so they will give the same answer:

$$Area = \frac{1}{2}ab\sin C$$

$$Area = \frac{1}{2}bc\sin A$$

$$Area = \frac{1}{2}ac\sin B$$

Pick any two and write them equal to each other, then rearranging lets us find a link between sides and angles:

$$\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A$$

$$ab\sin C = bc\sin A$$

$$a\sin C = c\sin A$$

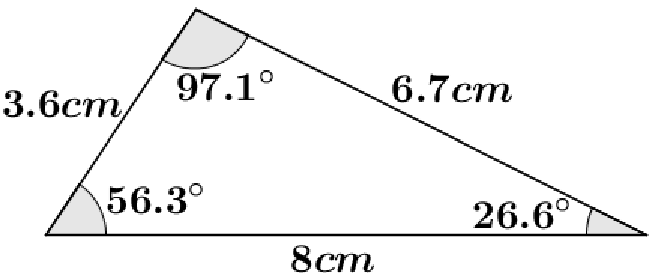
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Doing the same with other pairs of equations gets the full formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Checking with a real example

The triangle below is accurately drawn. Lengths and angles are correct to $1\ d.p.$



You can check that all three versions of the area formula give the same value for area.

Divide each side length by the sine of the angle opposite:

$$\frac{\mathbf{3.6}}{\mathbf{\sin 26.6}} =$$

$$\frac{\mathbf{6.7}}{\mathbf{\sin 56.3}} =$$

$$\frac{\mathbf{8}}{\mathbf{\sin 97.1}} =$$

These three should all be equal*

Small angles have a small sine, so the longer sides are opposite larger angles.

* Due to the fact that initial values were rounded, you'll get something like 8.04, 8.05 and 8.06.

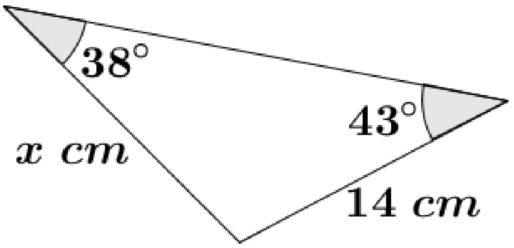
Using the Sine Rule

The sine rule can be used to find missing sides or angles whenever:

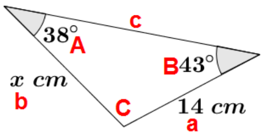
- You know a **side** and the **angle opposite** *and* **one other side or angle**.
- You want to find the **angle or side opposite the known one**.

Examples

1. Find the size of length x.



First, label the sides and angles:



Next, substitute into the formula:

$$\frac{14}{\sin 38} = \frac{x}{\sin 43}$$

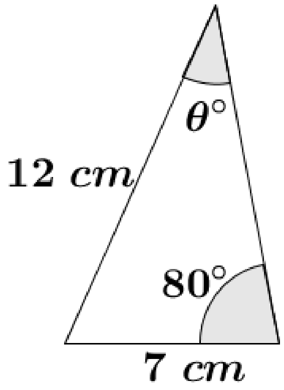
Then rearrange:

$$x = \frac{14 \sin 43}{\sin 38}$$

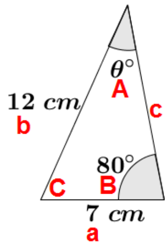
Lastly, solve:

$$x = \mathbf{15.5\ cm}$$

2. Find the size of angle θ.



First, label the sides and angles:



Next, substitute into the formula:

$$\frac{7}{\sin\theta} = \frac{12}{\sin 80}$$

Then rearrange:

$$\sin\theta = \frac{7 \sin 80}{12}$$

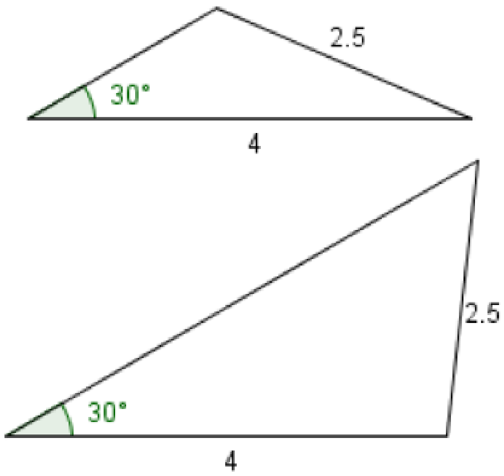
Lastly, solve:

$$\theta = \sin^{-1} 0.574 \ldots$$

$$\theta = \mathbf{35.1°}$$

Bonus Info: The Ambiguous Case

If you're trying to find an unknown angle, sometimes there are **two** equally valid alternative solutions, meaning the original information was ambiguous:



The two triangles to the left *both* have sides of length $4cm$ and $2.5cm$, with a $30°$ angle opposite the $2.5cm$ side. Using Sine Rule to find the top angle (opposite the $4cm$ side) gives:

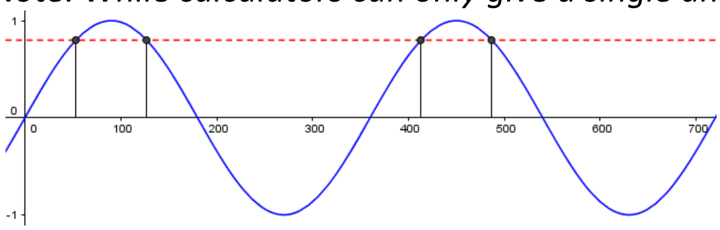
$$\frac{2.5}{\sin 30} = \frac{4}{\sin\theta} \Longrightarrow \sin\theta = \frac{4 \sin 30}{2.5}$$

$$\sin\theta = 0.8 \Longrightarrow \theta = 53°$$

(this solution fits the second triangle)

But the complement of $53°$ (the angle $180 - 53 = 127°$) also solves $\sin\theta = 0.8$, and *this* solution fits the first triangle.

Note: While calculators can only give a single answer, the graph shows multiple solutions:



The sine wave is mirrored at $90°$, and repeats itself every $360°$. Although we initially only used sine for acute angles, it works for any, *including obtuse, reflex, and $< 0°$ or $> 360°$.*