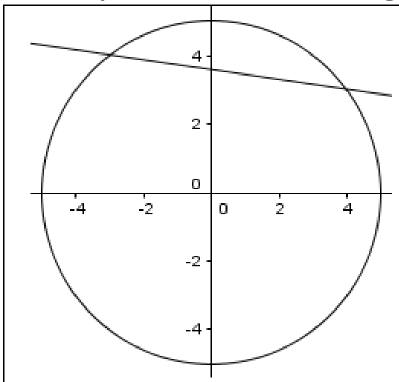
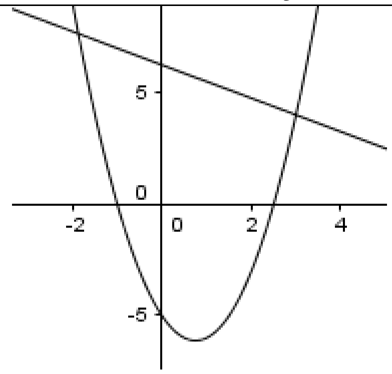
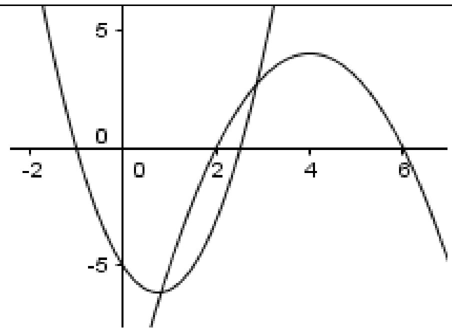
Simultaneous Equations: Line & Curve

Although elimination is often the most efficient method for 'nice' linear simultaneous equations, more complicated equations require substitution.

For equations involving curves, there are often multiple solutions:

 Circle & line: 0, 1 or 2 solutions	 Quadratic & line: 0, 1 or 2 solutions	 Quadratic & quadratic: 0, 1 or 2 solutions *(you won't see these at GCSE)*
Don't mix up the x and y coordinates: $(1,2)$ *and* $(6,7)$ *are not the same points as* $(1,7)$ *and* $(6,2)$		Note: All of these examples are solved in the same way: Rearrange one equation to substitute into the other.

Solve simultaneously:

$$\boldsymbol{x^2 + y^2 = 5}$$
$$\boldsymbol{y = 3x + 1}$$

Example

Substitute for y, from the linear into the quadratic:

$$x^2 + (3x+1)^2 = 5$$

Simplify, rearrange and solve the resulting quadratic:

$$x^2 + 9x^2 + 6x + 1 = 5$$
$$10x^2 + 6x - 4 = 0 \implies 5x^2 + 3x - 2 = 0$$
$$(5x-2)(x+1) = 0 \implies x = 0.4 \ \ or \ \ x = -1$$

Substitute each x value back into the linear equation:

$$x = 0.4 \implies y = 3(0.4) + 1 = 2.2 \implies \mathbf{(0.4, 2.2)}$$
$$x = -1 \implies y = 3(-1) + 1 = -2 \implies \mathbf{(-1, -2)}$$

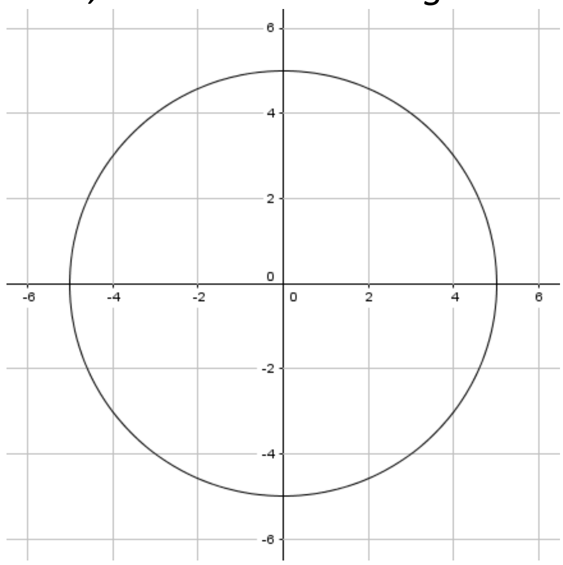
The circle shown has equation $x^2 + y^2 = 25$. Find any points of intersection with the lines below. *Which never crosses the circle, and which is a tangent?*

1) $x = -4$

2) $4y = -3x$

3) $x + y = 8$

4) $3x + 4y = 25$



Simultaneous Equations: Line & Curve SOLUTIONS

1)

$$x = -4 \quad \Longrightarrow \quad (-4)^2 + y^2 = 25$$

$$16 + y^2 = 25$$

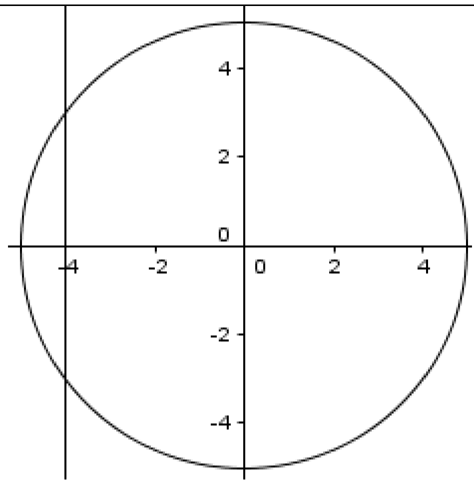
$$y^2 = 9$$

$$y = 3 \;\; or \;\; y = -3$$

Substitute back to find corresponding values of x:
(note: $x = -4$ anywhere on the line!)

$$y = 3 \;\Longrightarrow\; x = -4 \;\Longrightarrow\; (\mathbf{-4, 3})$$

$$y = -3 \;\Longrightarrow\; x = -4 \;\Longrightarrow\; (\mathbf{-4, -3})$$



The line is vertical, so the crossing points are symmetrical about the x axis.

2.

$$y = -\frac{3}{4}x \;\Longrightarrow\; x^2 + \left(-\frac{3}{4}x\right)^2 = 25$$

$$x^2 + \frac{9}{16}x^2 = 25$$

$$\frac{25}{16}x^2 = 25$$

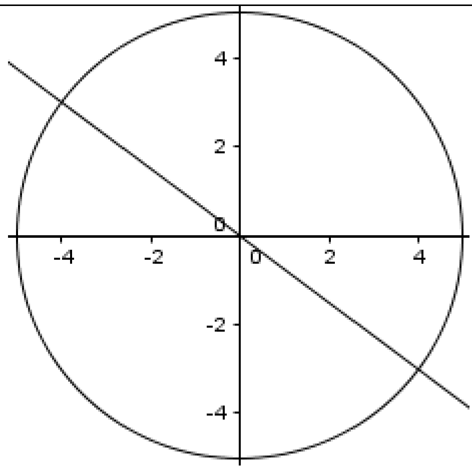
$$x^2 = 16$$

$$x = 4 \;\; or \;\; x = -4$$

Substitute back to find corresponding values of y:

$$x = 4 \;\Longrightarrow\; y = -\frac{3}{4}(4) = -3 \;\Longrightarrow\; (\mathbf{4, -3})$$

$$x = -4 \;\Longrightarrow\; y = -\frac{3}{4}(-4) = 3 \;\Longrightarrow\; (\mathbf{-4, 3})$$



The line goes through the origin, so each point is a 180° rotation about the origin from the other.

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3)

$$x = 8 - y \implies (8-y)^2 + y^2 = 25$$

$$64 - 16y + y^2 + y^2 = 25$$

$$2y^2 - 16y + 39 = 0$$

$$b^2 - 4ac = (-16)^2 - 4(2)(39) = -56 < 0$$

$$b^2 - 4ac < 0 \implies \boldsymbol{No\ Solutions}$$

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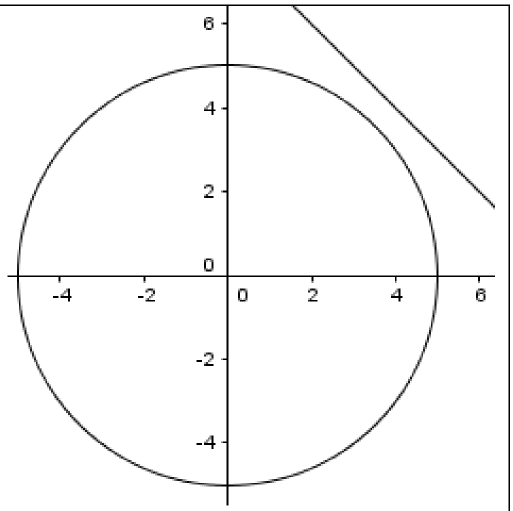

The line never touches the circle.

This is the geometric interpretation of finding no solutions to the resulting quadratic equation.

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4)

$$y = \frac{25 - 3x}{4} \implies x^2 + \left(\frac{25-3x}{4}\right)^2 = 25$$

$$x^2 + \frac{(25-3x)^2}{16} = 25$$

$$16x^2 + (25-3x)^2 = 400$$

$$16x^2 + 625 - 150x + 9x^2 = 400$$

$$25x^2 - 150x + 225 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x = 3$$

Substitute back to find corresponding value of y:

$$x = 3 \implies y = \frac{25 - 3(3)}{4} = 4 \implies (\mathbf{3}, \mathbf{4})$$

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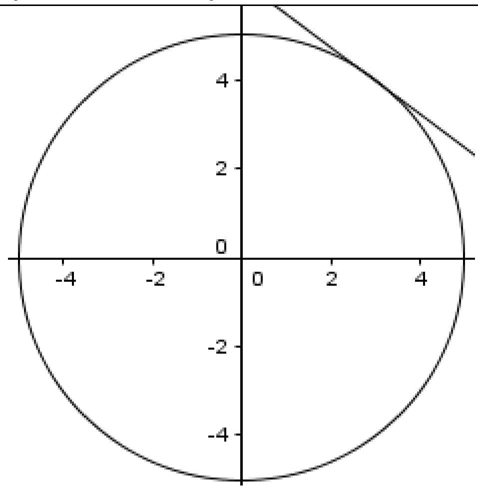

The line is a tangent line, only meeting the circle at one point.

This is the geometric interpretation of a repeated root (exactly one solution) when solving the quadratic equation.

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