Imaginary Numbers

"Mental torture" Girolamo Cardano (1545)

Many students, when first encountering imaginary numbers will make this argument:

"But imaginary numbers aren't real, so there's no point in learning about them."

True, they're not real. But why does that mean they're not worth learning about? Are fictional stories not worth reading? Is our imagination worthless? The telephone wasn't real 200 years ago, but it never would have become real without first existing as an imaginary telephone…

The catch is that *no* numbers are real. At least, not in the sense that beans and sausages are real:

Problem	**Invention**	**Objection**	**Mathematician's answer**	**Applications**
$5 \div 2 = ?$	Fractions $\frac{a}{b}$	"You can't have more than 1 object but less than 2. And you can't share 3 objects fairly between 2 people."	"What if you could? It would take two 'halves' to make 1. It's the opposite of 'lots of'."	Measurement and proportion.
$5 - 5 = ?$	Zero 0	"You can't have a number lower than any positive number. If you don't have anything, how can you measure it at all?"	"What if you could? It would be a number that wouldn't change other numbers when you add it on or take it off."	Place holder in place value number systems, improving hugely on Egyptian or Roman numerals.
$5 - 6 = ?$	Negatives $-a$	"You can't have numbers smaller than 0. Even if 'nothing' is a number, how can you have less than that?"	"What if you could? It would have the opposite effect to the equivalent positive number, so a -5 would cancel out a 5. We could easily extend the rules of multiplication and division."	Debt and borrowing, height and depth, temperature. Describing both direction and magnitude.
$\sqrt{5} = ?$	Irrationals $\sqrt{2}$ π e	"You can't have a number that is impossible to make though division. There are an unlimited number of fractions between any two numbers. Surely that's enough."	"What if there were numbers between the infinitely dense rationals? It would explain why $\sqrt{2}$ can never become a whole number no matter what whole number you multiply it by. It still gives a nice answer when you square it."	Pythagoras' theorem (surds), circle geometry (π), Fibonacci (golden ratio), exponential growth, computation and approximation.
$\sqrt{-5} = ?$	Imaginary numbers i	"You can't square-root a negative number. Any number I multiply by itself will yield either a positive answer or 0."	"What if you could? Imagine a number that, when squared, gives -1. How would a number system involving them work? Multiplying by i does half the job of multiplying by -1 so maybe it has something to do with direction as well."	Geometry, trigonometry, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography, relativity, fractals.

Every number system we use, from the natural numbers (1, 2, 3, …) to the integers (…, -1, 0, 1, 2, …), the rationals (… , $\frac{1}{2}$, $\frac{-3}{4}$, $\frac{22}{7}$, …) and the reals ($\ldots, 5, \sqrt{2}, \frac{7}{9}, e, \pi, \frac{\pi}{2}, \ldots$) – not to mention the hyperreals or transfinites – are invented. Initially this was to solve very down-to-earth problems such as dividing land or keeping track of monetary transactions, but increasingly done just for fun, by mathematicians. Strangely enough, no matter how abstract and 'recreational' mathematics becomes, there are always physicists, computer scientists, astronomers, engineers, economists, graphic designers, etc who will find all sorts of applications for it.

To be fair, it was many years before even mathematicians were happy to use imaginary numbers (hence the derogatory name). But they got used to the idea eventually, and now, just like every other 'imaginary' number, the mathematics that develops from the use of complex numbers is used every day to find 'real' solutions to problems.

Are letters real? Or is the alphabet just a collective delusion? Would that make it any less useful?
(Remember, what you see here is no more than marks on a page which represent the idea of a letter like digits represent the idea of a number…)