# Introduction to Integration

Understanding Integration

If *differentiation* gives a meaningful answer to $0 \div 0$ (gradient of a curve), then *integration* gives a meaningful answer to $0 \times \infty$ (area under a curve).

Integration is the process of adding up an infinite number of infinitesimally small amounts. By considering what happens as small pieces shrink to nothing (and the number of them rises towards infinity), we can find exact answers to otherwise impossible questions.

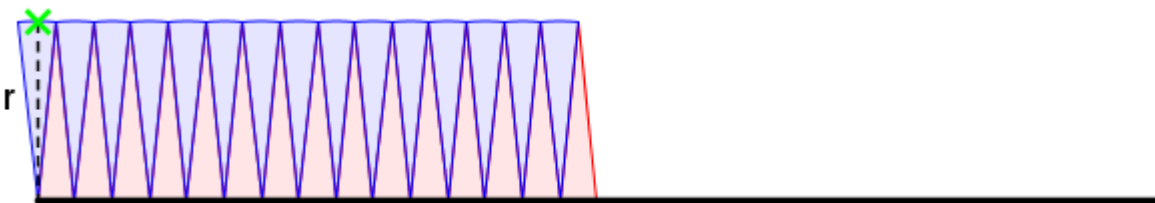**Example:** Finding the area of a circle.

Step 1: Unwrap the circumference. This must be of length $2\pi r$.
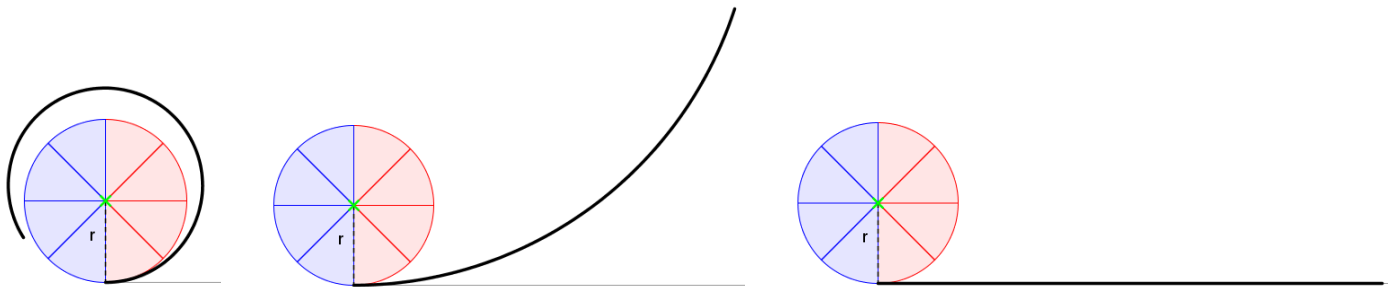
Step 2: Split the circle into a number of sectors and rearrange them.
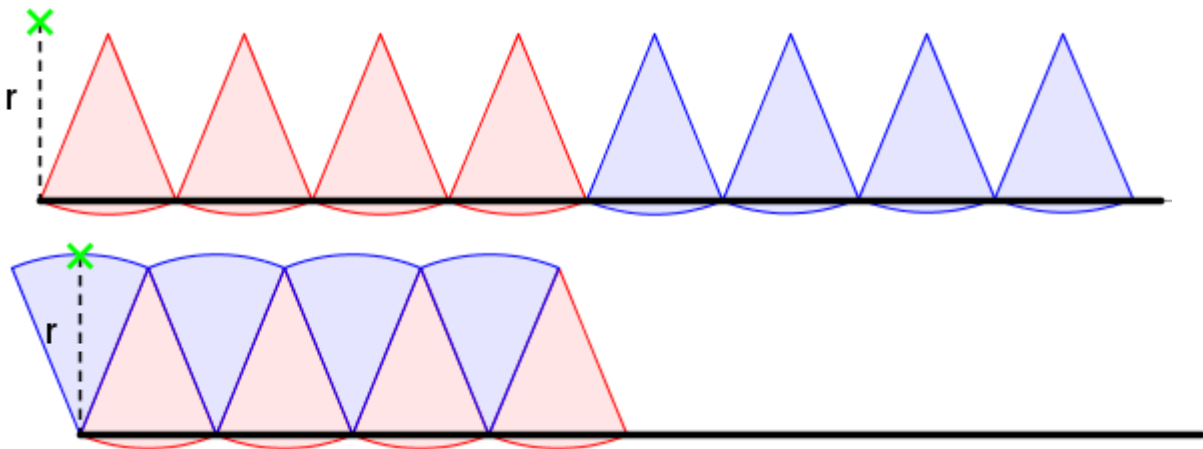
Step 3: Increase the number of slices used (they will get smaller).

Step 4: Notice that as the number of slices increases (and their size decreases), the shape increasingly resembles a rectangle.

This rectangle has height r and width πr, giving an area of:

$$A = r \times \pi r = \boldsymbol{\pi r^2}$$

Reversing Differentiation

Integration is the opposite of differentiation
The integral is the opposite of the derivative

Differentiate   Integrate

$f(x)$ = $\int F(x)$	$f'(x)$ = $F(x)$
Eg: x	1
(The derivative of x is 1, so the integral of 1 is x)	
x^2	
x^3	
x^4	
	x
	x^2
	x^3

Fill in the blanks, then try to find a general rule:

The integral of x^n is:

The Constant of Integration

Dave spends £300, earns £500, then spends a further £100.
How much money does he have now?

A quadratic differentiates to $2x - 3$.
Which of these could have been my quadratic?

a) $x^2 - 3x$ b) $2x^2 - 6x$ c) $x^2 - 3x + 3$ d) $(x + 2)(x - 5)$

Imagine trying to work out how much money someone has by keeping track of their income and their spending. You can tell a lot from this (how much more they have this month compared to last month, for instance), but unless you see the actual bank balance at some point you have no way of knowing the amount.

Since **information is lost** when we differentiate, it is not possible, without additional information, to perfectly reverse the process.	*You have come across similar ideas before: If I square a number and get 9, you can't know for sure whether my original number was 3 or -3.*

The only information lost when you differentiate is the constant term (number). Integration **does** tell us the **shape** of our original function. The only thing it **can't** tell us is the actual height of the graph.

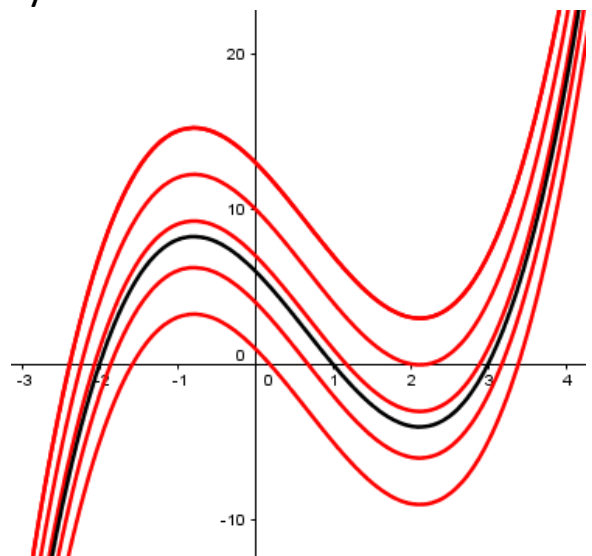The curves below all belong to the same 'family' of functions.

They are *all* of the form:
$$f(x) = x^3 - 2x^2 - 5x + C$$

The only unknown is C. The greater the value of C, the higher up the curve.

All of these curves have exactly the same gradient function:
$$f'(x) = 3x^2 - 4x - 5$$

The Format of Integration Questions

Since integration is the reverse of differentiation, often a question will provide you with a gradient function, $\frac{dy}{dx}$ or $f'(x)$ and ask for the 'original' function, y or $f(x)$. Alternatively, since integration is actually more powerful than simply the reverse of differentiation, it has its own symbol and notation. "The integral of ... with respect to x" is written as $\int ... \, dx$:

All of the following are asking you to do the same thing. Your notation should reflect that used in the question, but the process you follow is essentially the same:

Question	Solution
Integrate $3x^2 - 4x$ with respect to x.	$x^3 - 2x^2 + C$
$\frac{dy}{dx} = 3x^2 - 4x$. Find y.	$y = x^3 - 2x^2 + C$
$f'(x) = 3x^2 - 4x$. Find $f(x)$.	$f(x) = x^3 - 2x^2 + C$
Find: $\int 3x^2 - 4x \, dx$.	$x^3 - 2x^2 + C$
Find the integral of $3x^2 - 4x$ with respect to x.	$x^3 - 2x^2 + C$
A curve has gradient $3x^2 - 4x$. Find the equation of the curve.	$y = x^3 - 2x^2 + C$

Finding the constant

With no further information, your answers must include an arbitrary constant: $+ C$.
However, if you are given additional information, you can calculate C.

Eg: A curve with gradient $\frac{dy}{dx} = 3x^2 - 4x$ goes through the point $(1,4)$. Find y.

Integrate and write with $+ C$:
$$y = x^3 - 2x^2 + C$$
Substitute in the known values:
$$y = 4 \ when \ x = 1 \implies 4 = (1)^3 - 2(1)^2 + C \implies 5 = C$$
Rewrite now you know C:
$$y = x^3 - 2x^2 + 5$$

Definite Integration

Indefinite integration gives a function, usually with a $+ C$ at the end.
Definite integration gives a specific value – it is equivalent to the area between two limits.

Eg: Find the value of $\int_1^4 3x^2 - 4x \, dx$.
Integrate, then substitute in the two limit values and find the difference:
$$\int_1^4 3x^2 - 4x \, dx = [x^3 - 2x^2]_1^4 = \{4^3 - 2(4)^2\} - \{1^3 - 2(1)^2\} = \{32\} - \{-1\} = \mathbf{33}$$

Note: $+ C$ is not necessary with definite integration; if you include it, it simply cancels out.

Finding Area Using Integration

While we can use integration to reverse differentiation, it actually means a lot more than that. Integration, at its most fundamental, is about adding things up.

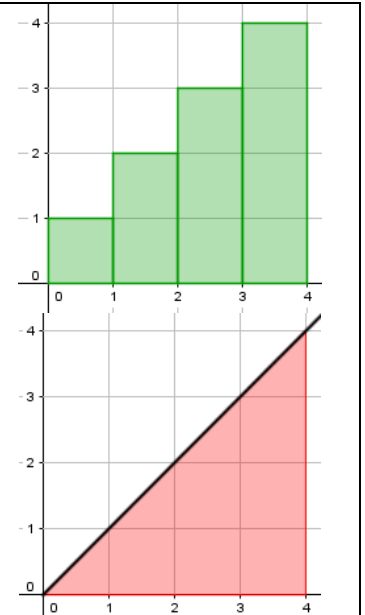But instead of adding a finite number of *discrete* values, like this summation function does…

$$\sum_{r=1}^{4} r = 1 + 2 + 3 + 4 = 10$$

(The Greek letter Σ (sigma) means 'sum').

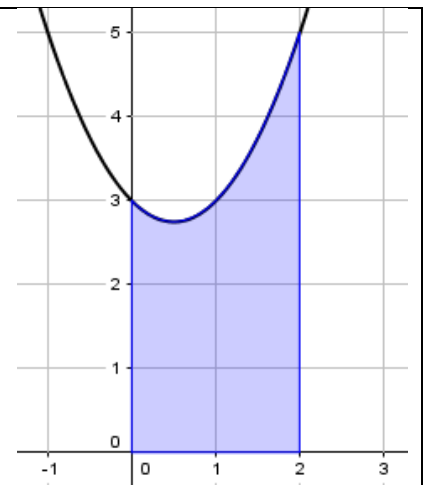… integration gives us a way to add up an infinite *continuum* of values:

$$\int_{0}^{4} x\, dx = 8$$

(The Medieval letter ∫ (long s) also means 'sum').

More generally, the **area under any curve** can be calculated by **integrating** the function between two limits. This means finding the difference between the value of the integral function at the two limits:

$$\int_{0}^{2} x^2 - x + 3 \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + 3x \right]_{0}^{2}$$

$$= \left\{ \frac{(2)^3}{3} - \frac{(2)^2}{2} + 3(2) \right\} - \left\{ \frac{(0)^3}{3} - \frac{(0)^2}{2} + 3(0) \right\} = \frac{20}{3}$$
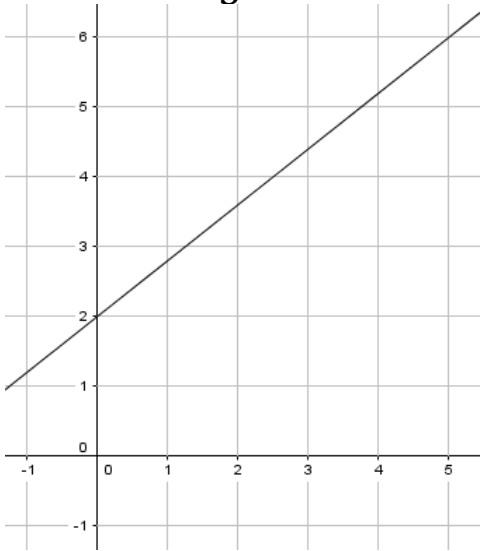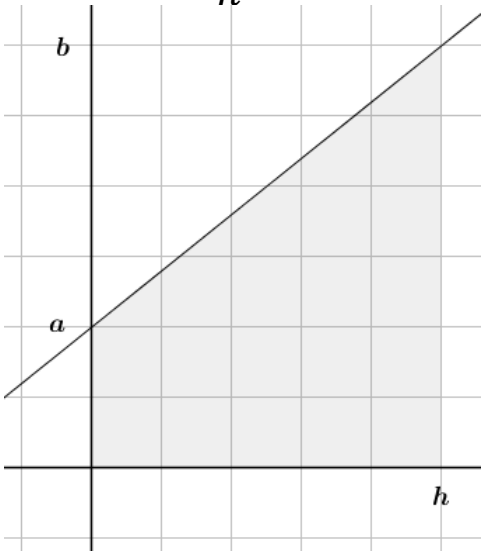
Caution: If the curve is below the x-axis, the integral will be negative. The *size* of the integral will give the area.

Terminology: When a function is integrated between two limits, the process is called *definite integration* as opposed to general integration without limits, which is known as *indefinite integration* and involves an arbitrary constant, C.
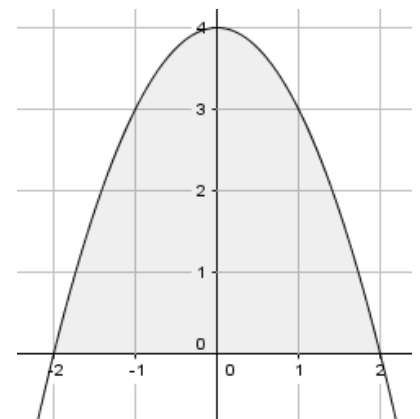
Area of common shapes

Integration has given us, in one go, a way to find the area of any shape whose boundary can be described by an integrable function. To convince yourself that this is consistent with finding area the basic way, consider a simple shape:
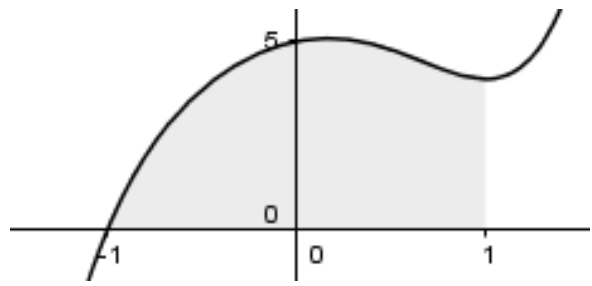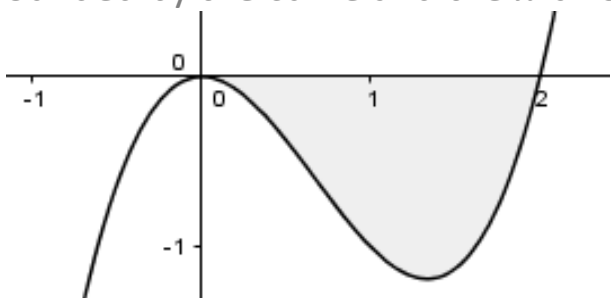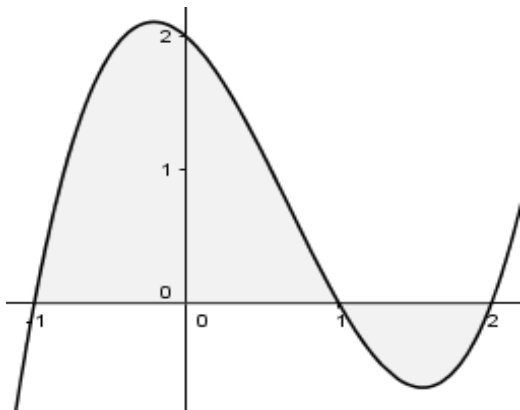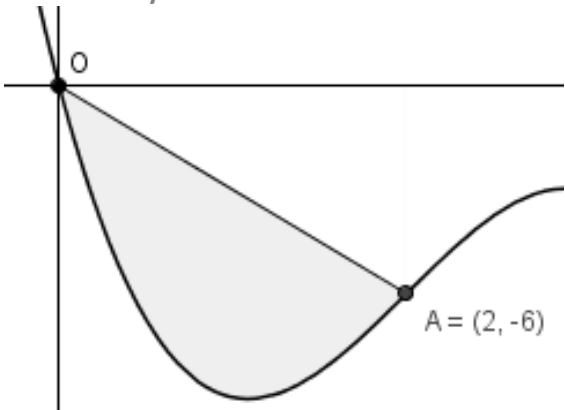
Area of a trapezium using integration

Specific Example	General Case (to find a formula)
The area of a trapezium with apex and base 2 and 6 respectively, and height 5 is equivalent to the area under: $$y = \frac{4}{5}x + 2$$ between 0 and 5. $$Area = \int_0^5 \frac{4}{5}x + 2 \; dx$$ $$= \left[\frac{2}{5}x^2 + 2x\right]_0^5$$ $$= \left\{\frac{2}{5}(5)^2 + 2(5)\right\} - \{0\} = \mathbf{20}$$	The area of a trapezium with apex and base a and b respectively, and height h is equivalent to the area under: $$y = \frac{b - a}{h}x + a$$ between 0 and h. $$Area = \int_0^h \frac{b - a}{h}x + a \; dx$$ $$= \left[\frac{b - a}{2h}x^2 + ax\right]_0^h$$ $$= \left\{\frac{b - a}{2h}h^2 + ah\right\} - \{0\} = \frac{\mathbf{a + b}}{\mathbf{2}}\mathbf{h}$$

Integration is wasted on simple shapes like trapezia. It isn't the most efficient way to find a formula for such simple shapes, but since it works for *any* polynomial curve, you can use it to show that, for instance, the area enclosed by the parabolic curve $y = a^2 - x^2$ is always equal to $\frac{4}{3}a^3$.

Integration Area Checklist

Skill	Example
1. Area under a curve; positive I can use definite integration to find the area between a curve and the x-axis for everywhere-positive intervals.	Find the area bounded by the curve, the x-axis and the line $x = 1$: $y = x^5 - 3x^2 + x + 5$
2. Area under a curve; negative I can use definite integration to find the area bounded by a curve and the x-axis for everywhere-negative intervals, interpreting my answers appropriately to take into account the sign of the integral.	Find the area of the finite region bounded by the curve and the x-axis: $y = x^2(x - 2)$
3. Area under a curve; mixed I can use definite integration, reading graphs or solving equations as necessary, to find the area enclosed by a curve and the x-axis for a combination of positive and negative intervals, taking into account the sign of the respective integrals.	Find the total area of the two shaded regions shown below: $y = (x - 2)(x + 1)(x - 1)$
4. Area between a curve and a line I can use definite integration, along with rules of area for rectangles, triangles and trapezia, to find the area enclosed by a curve and a line.	Find the area of the shaded region, enclosed by the curve and the line. $A = (2, -6)$ $y = 12x^2 - 19x - 2x^3$

Area Under Curves Exam Questions

1.

The curve with equation $y = x^3 - 3x^2 - 6x + 8$ is sketched below.
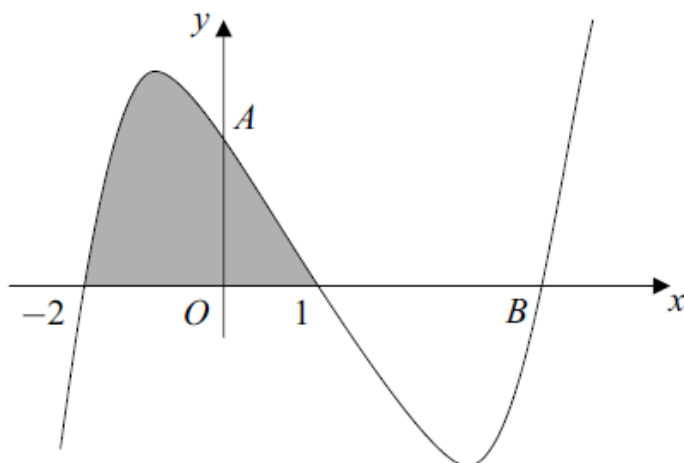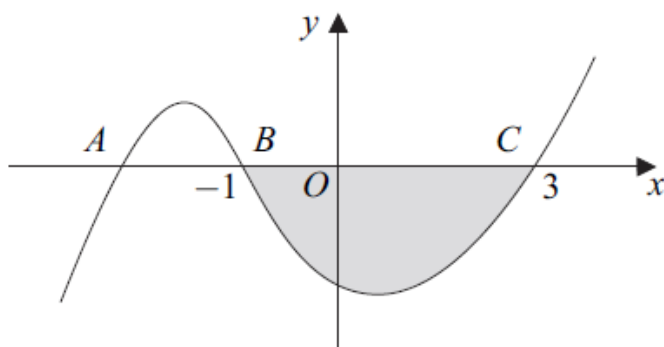

(i)   Find $\int (x^3 - 3x^2 - 6x + 8)\, dx$. *(4 marks)*

(ii) Hence find the area of the shaded region bounded by the curve and the x-axis.
(3 marks)

2.

The curve with equation $y = x^3 - 7x - 6$ is sketched below.

The curve cuts the x-axis at the point A and the points $B(-1, 0)$ and $C(3, 0)$.

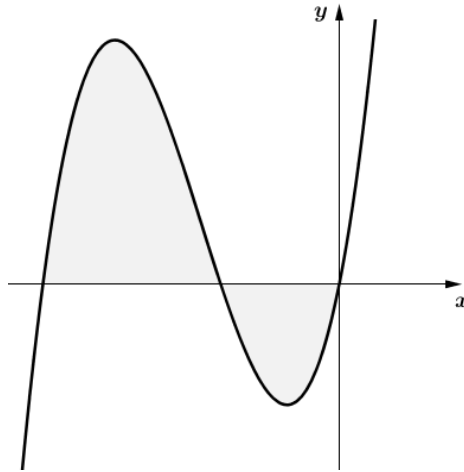(i)   Find $\int_{-1}^{3} (x^3 - 7x - 6)\, dx$. *(5 marks)*

(ii) Hence find the area of the shaded region bounded by the curve $y = x^3 - 7x - 6$
and the x-axis between B and C. *(1 mark)*

3.

The curve with equation $y = x^3 + 4x^2 + 3x$ is sketched below.

(i) Express $x^3 + 4x^2 + 3x$ in the form $x(x + p)(x + q)$ where p and q are integers.

(2 marks)

(ii) State the x-coordinates of the three points where the curve intersects the x-axis.

(1 mark)

(iii) Find the total area of the shaded region bounded by the curve and the x-axis.

(6 marks)

4.

The curve C with equation $y = x^2 - 3x + 5$ and the straight line $y = x + 5$ intersect at the point $A(0, 5)$ and at the point B, as shown in the diagram below.
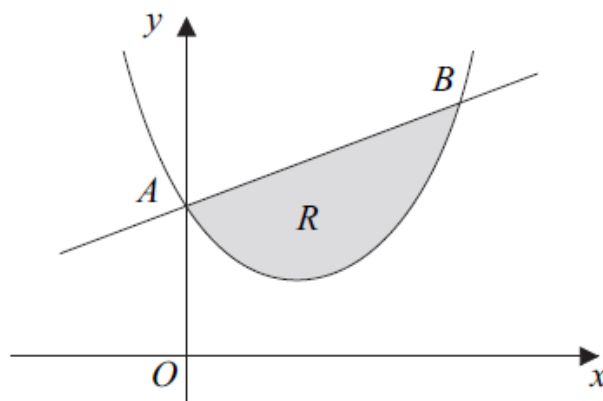
(i) Find the coordinates of the point B. *(3 marks)*

(ii) Find $\int (x^2 - 3x + 5)\, dx$. *(3 marks)*

(iii) Find the area of the shaded region R bounded by the curve C and the line segment AB.

(4 marks)

Integration Challenge Question

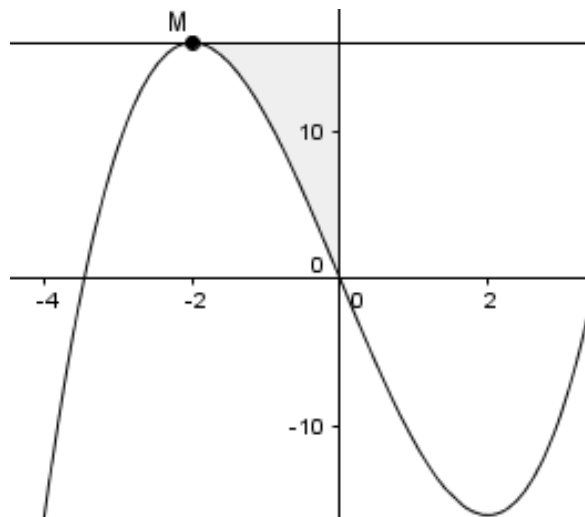The curve with equation $y = x^3 - 12x$ is shown below:
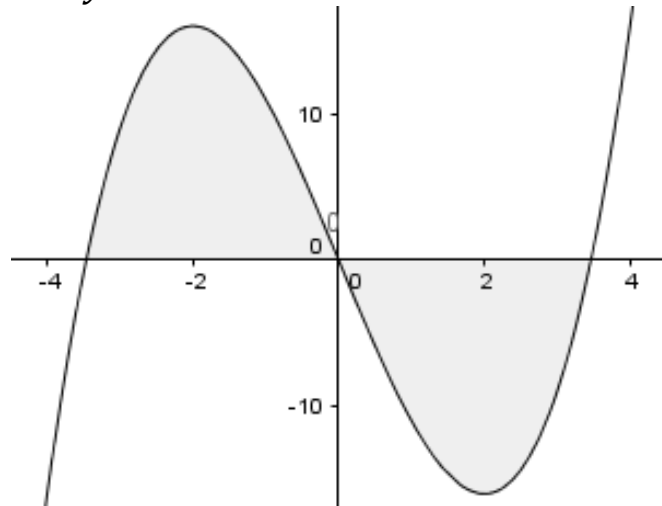
The curve crosses the x-axis at the origin and two other points.

a) Show that the x-coordinates of the other two points are $2\sqrt{3}$ and $-2\sqrt{3}$.

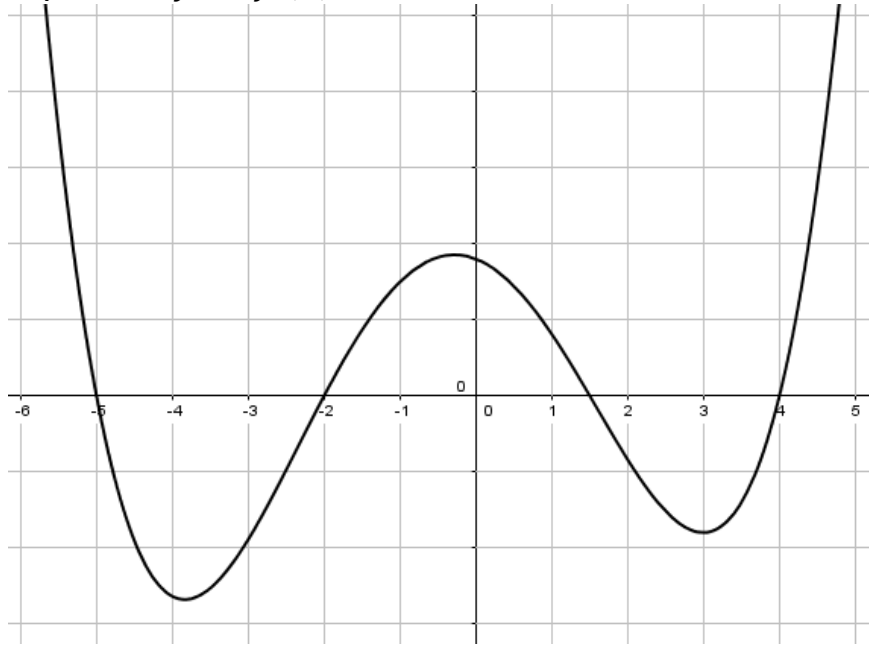b) Calculate the value of the integral:

$$\int_{-2\sqrt{3}}^{2\sqrt{3}} x^3 - 12x \ dx$$

c) What does this value tell you about the area of the two shaded regions?

d) By forming suitable integrals, find the total area of the shaded regions.

e) Find the coordinates of the maximum point, M.

f) Calculate the area of the shaded region enclosed by the curve, the tangent at M and the y-axis:

Combining Areas with Integration

The curve with equation $y = f(x)$ is shown below.

The curve crosses the x-axis at $(-5,0)$, $(-2,0)$, $(1.5,0)$ and $(4,0)$.

Using the information provided below, answer the questions that follow:

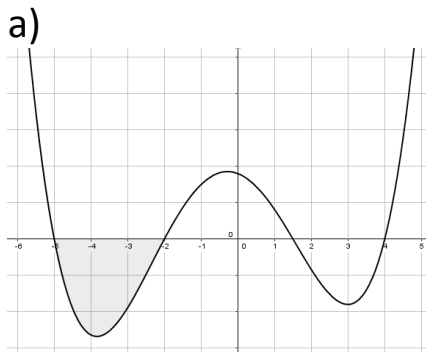$$\int_{-5}^{-2} f(x)\ dx = -5 \qquad \int_{-5}^{1.5} f(x)\ dx = -1 \qquad \int_{-2}^{4} f(x)\ dx = 1$$

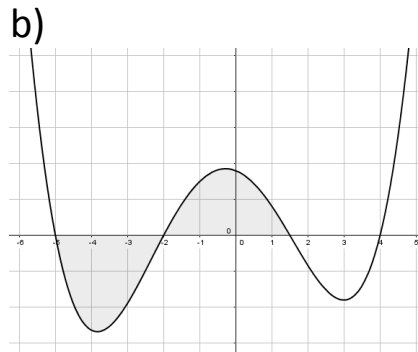1. Find the value of the following integrals:

a)
$$\int_{-2}^{1.5} f(x)\ dx =$$

b)
$$\int_{1.5}^{4} f(x)\ dx =$$

c)
$$\int_{-5}^{4} f(x)\ dx =$$

2. Find the total area of the shaded region in each diagram:

a)

$Area =$

b)

$Area =$

c)
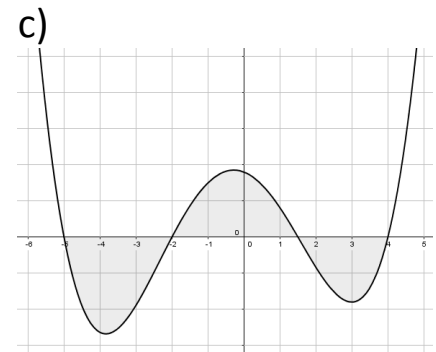
$Area =$

Recall that integration calculates the *area below a curve*, meaning if the curve is actually *below* the x-axis in certain places, the integral will be negative.
In some cases we want the *area* (eg total distance travelled on a journey), and in some cases the *integral* (eventual distance away from starting point).

Applications of Integration

As glacial meltwater feeds into a river, the flow rate increases by 600 litres per hour per hour, from the usual rate of 2000 litres per hour.

The **flow rate**, f, at time t hours after midnight is:
$$f = 600t + 2000$$

The total **volume** of water that passes a certain point between 1am and 5am is given by:
$$\int_1^5 600t + 2000 \ dt$$

As a car brakes to reduce its speed from $30m/s$ (approx 60mph) to $15m/s$ (around 30mph), its **speed**, v, at time t is given by:
$$v = 30 - 5t$$

The total **distance** the car travels during this time is given by:
$$\int_0^3 30 - 5t \ dt$$

The **force**, F Newtons, exerted by a bungee rope on a jumper when stretched a distance x metres is:
$$F = 800x$$

The **elastic energy** stored in the rope between becoming taut and reaching maximum length is:
$$\int_0^{30} 800x \ dx$$

There are many, many more examples, but most involve more complicated functions (economic growth uses e^x and planetary motion involves $\sin x$ and $\cos x$).

And this is still just 2-D. Integration can be adapted to find volumes (Core 3), and will eventually be used to solve differential equations (Core 4) which are a vital element of many physical and financial calculations.

Reversing Differentiation SOLUTIONS

Integration is the opposite of differentiation
The integral is the opposite of the derivative

Differentiate   Integrate

$f(x)$ = $\int F(x)$	$f'(x)$ = $F(x)$
Eg: x	1
(The derivative of x is 1, so the integral of 1 is x)	
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
$\dfrac{x^2}{2}$	x
$\dfrac{x^3}{3}$	x^2
$\dfrac{x^4}{4}$	x^3

Fill in the blanks, then try to find a general rule:

The integral of x^n is: $\dfrac{x^{n+1}}{n+1}$

Area Under Curves Exam Questions SOLUTIONS

1.

i.

$$\int x^3 - 3x^2 - 6x + 8 \; dx = \frac{x^4}{4} - x^3 - 3x^2 + 8x + C$$

ii.

$$\int_{-2}^{1} x^3 - 3x^2 - 6x + 8 \; dx = \left[\frac{x^4}{4} - x^3 - 3x^2 + 8x\right]_{-2}^{1}$$

$$= \left(\frac{1^4}{4} - 1^3 - 3(1^2) + 8(1)\right) - \left(\frac{(-2)^4}{4} - (-2)^3 - 3(-2)^2 + 8(-2)\right) = (4.25) - (-16) = \mathbf{20.25}$$

2.

i.

$$\int_{-1}^{3} x^3 - 7x - 6 \; dx = \left[\frac{x^4}{4} - \frac{7x^2}{2} - 6x\right]_{-1}^{3}$$

$$= \left(\frac{3^4}{4} - \frac{7(3^2)}{2} - 6(3)\right) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^2}{2} - 6(-1)\right) = \left(-\frac{117}{4}\right) - \left(\frac{11}{4}\right) = \mathbf{-32}$$

ii.

$$Area = |-32| = \mathbf{32}$$

3.

i.

$$x^3 + 4x^2 + 3x = x(x^2 + 4x + 3) = \boldsymbol{x(x+1)(x+3)}$$

ii.

$$\boldsymbol{x = 0, \; x = -1} \text{ and } \boldsymbol{x = -3}$$

iii.

$$First \; section \; (above \; x - axis): \quad \int_{-3}^{-1} x^3 + 4x^2 + 3x \; dx = \left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2}\right]_{-3}^{-1}$$

$$= \left(\frac{(-1)^4}{4} + \frac{4(-1)^3}{3} + \frac{3(-1)^2}{2}\right) - \left(\frac{(-3)^4}{4} + \frac{4(-3)^3}{3} + \frac{3(-3)^2}{2}\right) = \left(\frac{5}{12}\right) - \left(-\frac{9}{4}\right) = \frac{32}{12} = \frac{8}{3}$$

$$Second \; section \; (below \; x - axis): \quad \int_{-1}^{0} x^3 + 4x^2 + 3x \; dx = \left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{0}$$

$$= \left(\frac{(0)^4}{4} + \frac{4(0)^3}{3} + \frac{3(0)^2}{2}\right) - \left(\frac{(-1)^4}{4} + \frac{4(-1)^3}{3} + \frac{3(-1)^2}{2}\right) = (0) - \left(\frac{5}{12}\right) = -\frac{5}{12}$$

$$Total \; area = \left|\frac{8}{3}\right| + \left|-\frac{5}{12}\right| = \frac{37}{12}$$

4.

i.

$$x^2 - 3x + 5 = x + 5 \implies x^2 - 4x = 0 \implies x(x - 4) = 0 \implies x = 0 \;\; or \;\; x = 4$$
$$x = 0 \;\; represents \; the \; point \; A: (0,5). \quad x = 4 \implies y = (4) + 5 = 9 \implies \boldsymbol{B: (4, 9)}$$

ii.
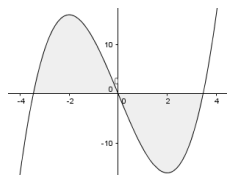
$$\int x^2 - 3x + 5 \; dx = \frac{x^3}{3} - \frac{3x^2}{2} + 5x + C$$

iii.

$$Area \; under \; line = \frac{a + b}{2} h = \frac{5 + 9}{2}(4) = 28$$

$$Area \; under \; curve = \int_{0}^{4} x^2 - 3x + 5 \; dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 5x\right]_{0}^{4} = \left(\frac{4^3}{3} - \frac{3(4^2)}{2} + 5(4)\right) - (0) = \frac{52}{3}$$

$$Area \; of \; region \; R = (Area \; under \; line) - (Area \; under \; curve) = 28 - \frac{52}{3} = \frac{32}{3}$$

Integration Challenge Question SOLUTIONS

The curve with equation $y = x^3 - 12x$ is shown below:

The curve crosses the x-axis at the origin and two other points.

a) Show that the x-coordinates of the other two points are $2\sqrt{3}$ and $-2\sqrt{3}$.

$$x^3 - 12x = 0 \implies x(x^2 - 12) = 0 \implies x = 0 \ or \ x^2 - 12 = 0$$
$$x^2 = 12 \implies x = \pm\sqrt{12} = \pm 2\sqrt{3}$$

b) Calculate the value of the integral:

$$\int_{-2\sqrt{3}}^{2\sqrt{3}} x^3 - 12x \ dx = \left[\frac{x^4}{4} - 6x^2\right]_{-2\sqrt{3}}^{2\sqrt{3}}$$

$$= \left\{\frac{(2\sqrt{3})^4}{4} - 6(2\sqrt{3})^2\right\} - \left\{\frac{(-2\sqrt{3})^4}{4} - 6(-2\sqrt{3})^2\right\} = \{-36\} - \{-36\} = \mathbf{0}$$

c) What does this value tell you about the area of the two shaded regions?

The area of the part above the x-axis is equal to that below the x-axis.

d) By forming suitable integrals, find the total area of the shaded regions.

$$\int_{-2\sqrt{3}}^{0} x^3 - 12x \ dx = \left[\frac{x^4}{4} - 6x^2\right]_{-2\sqrt{3}}^{0}$$

$$= \left\{\frac{(0)^4}{4} - 6(0)^2\right\} - \left\{\frac{(-2\sqrt{3})^4}{4} - 6(-2\sqrt{3})^2\right\} = \{0\} - \{-36\} = 36$$
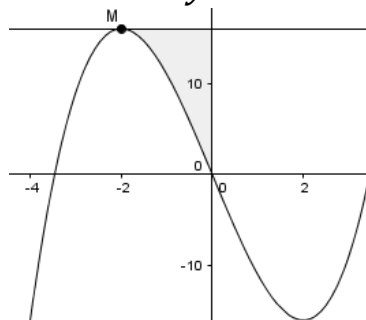
$$\implies Area = 2 \times 36 = \mathbf{72}$$

e) Find the coordinates of the maximum point, M.

$$\frac{dy}{dx} = 3x^2 - 12 \ \& \ Stat.pt \implies \frac{dy}{dx} = 0 \implies 3x^2 - 12 = 0 \implies x = \pm 2$$
$$(y)_{x=2} = 2^3 - 12(2) = -16 \ \& \ (y)_{x=-2} = (-2)^3 - 12(-2) = 16$$
$$\frac{d^2y}{dx^2} = 6x \implies \left(\frac{d^2y}{dx^2}\right)_{x=-2} = -12 < 0 \implies \mathbf{Maximum \ at \ (-2, 16)}$$

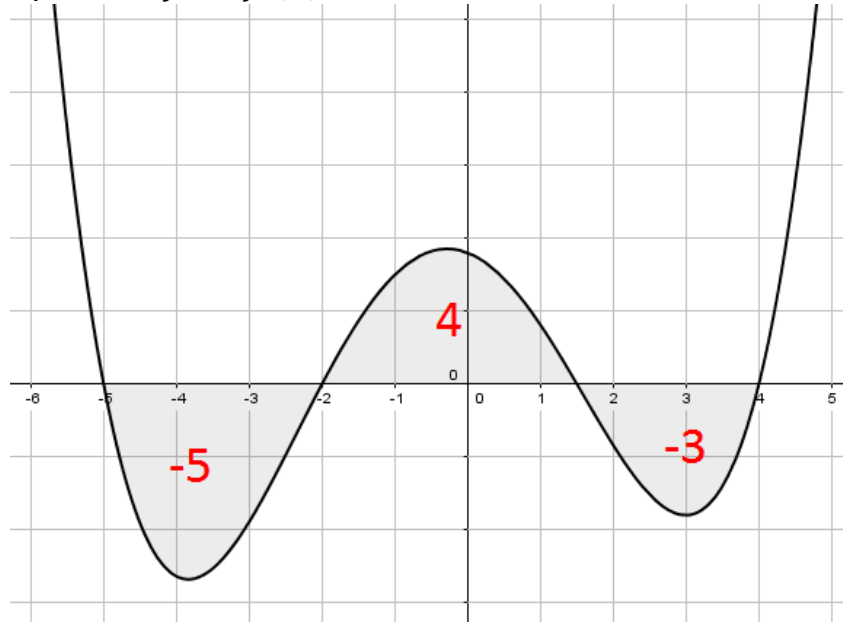f) Calculate the area of the shaded region enclosed by the curve, the tangent at M and the y-axis:

$$Area = 2 \times 16 - \int_{-2}^{0} x^3 - 12x \ dx$$

$$= 32 - (\{0\} - \{-20\}) = \mathbf{12}$$

Combining Areas with Integration SOLUTIONS

The curve with equation $y = f(x)$ is shown below.

The curve crosses the x-axis at $(-5,0)$, $(-2,0)$, $(1.5,0)$ and $(4,0)$.

Using the information provided below, answer the questions that follow:

$$\int_{-5}^{-2} f(x)\ dx = -5 \qquad \int_{-5}^{1.5} f(x)\ dx = -1 \qquad \int_{-2}^{4} f(x)\ dx = 1$$

1. Find the value of the following integrals:
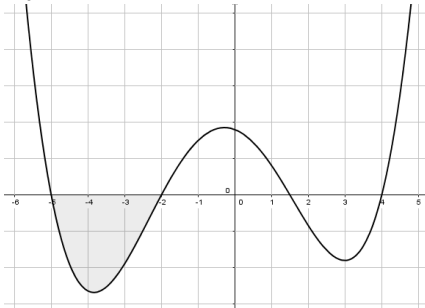
a)
$$\int_{-2}^{1.5} f(x)\ dx = 4$$

b)
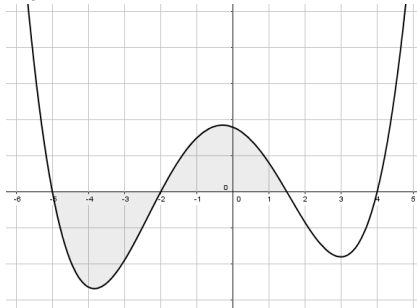$$\int_{1.5}^{4} f(x)\ dx = -3$$

c)
$$\int_{-5}^{4} f(x)\ dx = -4$$

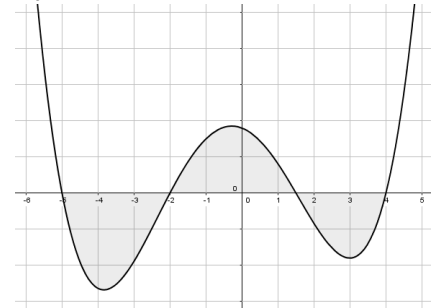2. Find the total area of the shaded region in each diagram:

a)

$Area = |-5| = 5$

b)

$Area = 5 + |4| = 9$

c)

$Area = 9 + |-3| = 12$

Recall that integration calculates the *area below a curve*, meaning if the curve is actually *below* the x-axis in certain places, the integral will be negative.

In some cases we want the *area* (eg total distance travelled on a journey), and in some cases the *integral* (eventual distance away from starting point).

Applications of Integration SOLUTIONS

As glacial meltwater feeds into a river, the flow rate increases by 600 litres per hour per hour, from the usual rate of 2000 litres per hour.

The **flow rate**, f, at time t hours after midnight is:
$$f = 600t + 2000$$

The total **volume** of water that passes a certain point between 1am and 5am is given by:
$$\int_1^5 600t + 2000 \ dt = [300t^2 + 2000t]_1^5$$
$$= (17500) - (2300) = \mathbf{15200 \ litres}$$

As a car brakes to reduce its speed from $30m/s$ (approx 60mph) to $15m/s$ (around 30mph), its **speed**, v, at time t is given by:
$$v = 30 - 5t$$

The total **distance** the car travels during this time is given by:
$$\int_0^3 30 - 5t \ dt = \left[30t - \frac{5t^2}{2}\right]_0^3$$
$$= \left(\frac{135}{2}\right) - (0) = \mathbf{67.5 \ metres}$$

The **force**, F Newtons, exerted by a bungee rope on a jumper when stretched a distance x metres is:
$$F = 800x$$

The **elastic energy** stored in the rope between becoming taut and reaching maximum length is:
$$\int_0^{30} 800x \ dx = [400x^2]_0^{30}$$
$$= (360000) - (0) = \mathbf{360,000 \ Joules}$$

There are many, many more examples, but most involve more complicated functions (economic growth uses e^x and planetary motion involves $\sin x$ and $\cos x$).

And this is still just 2-D. Integration can be adapted to find volumes (Core 3), and will eventually be used to solve differential equations (Core 4) which are a vital element of many physical and financial calculations.