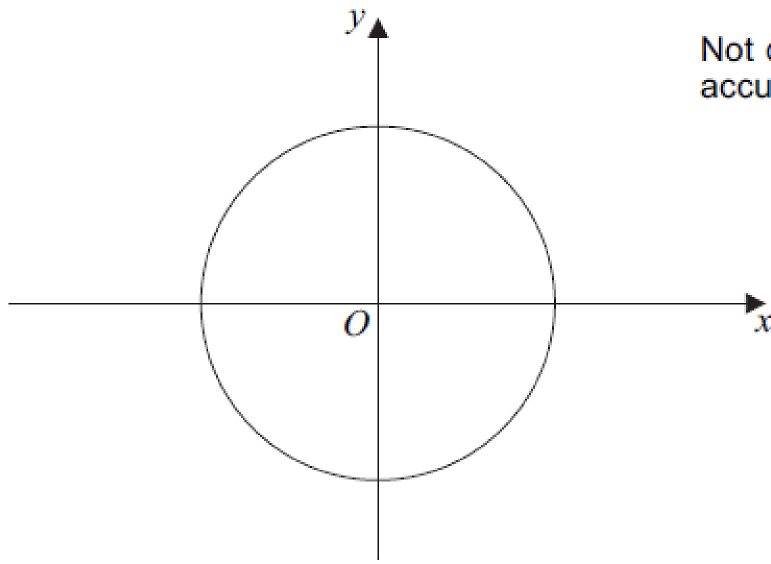
Further Maths Level 2 Exam Paper Solutions (June 2012)

Paper 2 – Calculator

1 Here is a sketch of the circle $x^2 + y^2 = 36$



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Work out the circumference of the circle.

....

....

....

Answer... *(3 marks)*

Use the equation to identify the radius, then apply the appropriate formula for circumference. *Recall that a circle, centred at $(0,0)$, of radius r, is represented by the equation $x^2 + y^2 = r^2$ and that the circumference of a circle is given by $C = 2\pi r$.*

$$x^2 + y^2 = 36 \quad \Longrightarrow \quad r = \sqrt{36} = 6$$

$$C = 2\pi(6) = 12\pi$$

$$\boldsymbol{Answer}: \quad \mathbf{12\pi}$$

Note: It would also be acceptable to evaluate your answer as a decimal (ie, 37.7 to 1 d.p.).

2 $y = 5x^3 - 4x^2$

Work out $\frac{dy}{dx}$.

$\frac{dy}{dx} =$ .... *(2 marks)*

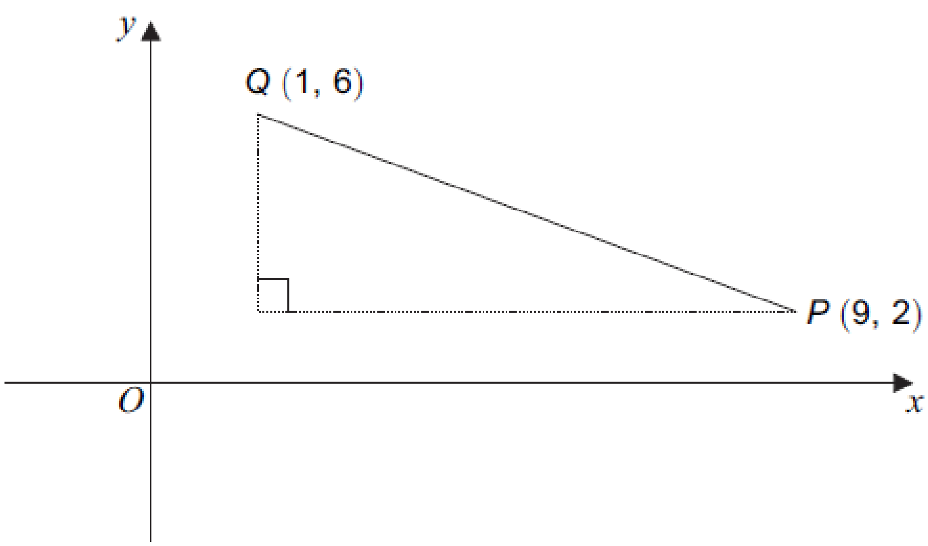
Differentiate the expression and simplify.

Recall that if $y = ax^n$ *then* $\frac{dy}{dx} = anx^{n-1}$.

$$y = 5x^3 - 4x^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 15x^2 - 8x$$

$$\boldsymbol{Answer}: \ \boldsymbol{\frac{dy}{dx} = 15x^2 - 8x}$$

3



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Work out the length of *PQ*.
Give your answer to 3 significant figures.

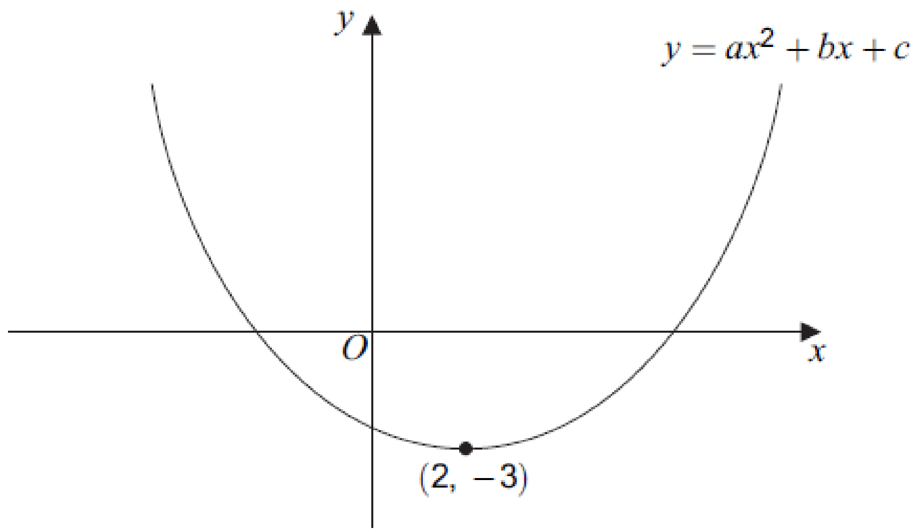
PQ = ... *(4 marks)*

Use the coordinates to calculate the length of the horizontal and vertical sides, then apply Pythagoras' theorem to calculate PQ.

$$PQ = \sqrt{(9-1)^2 + (2-6)^2} = \sqrt{80} = 4\sqrt{5} = 8.94 \ to \ 3 \ s.f.$$

$$\boldsymbol{Answer}: \quad \boldsymbol{PQ = 8.94 \ to \ 3 \ s.f.}$$

4 A sketch of $y = ax^2 + bx + c$ is shown.
The minimum point is $(2, -3)$.



For the sketch shown, circle the correct answer in each of the following.

4 (a) The value of a is

zero positive negative *(1 mark)*

Use your knowledge of quadratic graphs to interpret the shape and position.
Recall that, for large positive or large negative values of x, x^2 becomes very large, so that a positive value for a gives a ∪ shape while a negative value would give a ∩ shape. If $a = 0$, the graph is merely a straight line.

$$\boldsymbol{Answer}: \quad \boldsymbol{positive}$$

4 (b) The value of c is

zero positive negative *(1 mark)*

Use your knowledge of graphs and equations to identify key points of the graph.
Recall that c represents the y intercept for the graph, since when $x = 0$, $y = c$. Therefore it is necessary only to note where the curve cuts the y axis.

$$\boldsymbol{Answer}: \quad \boldsymbol{negative}$$

4 (c) The solutions of $ax^2 + bx + c = 0$ are

both zero | both positive | both negative | one positive and one negative

(1 mark)

Determine the nature of any solutions by identifying them on the graph.
Recall that $ax^2 + bx + c = 0$ is equivalent to finding where $y = ax^2 + bx + c$ crosses $y = 0$. That is, where the curve cuts the x axis.

$$\boldsymbol{Answer}\text{:} \quad \boldsymbol{one\ positive\ and\ one\ negative}$$

4 (d) The **number** of solutions of $ax^2 + bx + c = -6$ is

0 | 1 | 2 | 3 *(1 mark)*

Interpret the equation graphically and use the curve to analyse the situation.
Recall that $ax^2 + bx + c = -6$ is equivalent to finding where $y = ax^2 + bx + c$ crosses $y = -6$.

$$\boldsymbol{Answer}\text{:} \quad \mathbf{0}$$

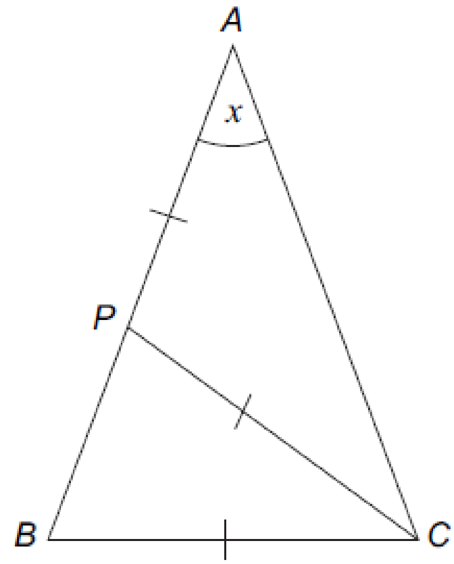
4 (e) The equation of the tangent to $y = ax^2 + bx + c$ at $(2, -3)$ is

$x = 2$ | $y = 2$ | $x = -3$ | $y = -3$ *(1 mark)*

Use the fact that a tangent line will touch at only one point on the curve, and will have the same gradient as the curve at that point.
Recall that at a minimum, the gradient of a curve is 0.

$$\boldsymbol{Answer}\text{:} \quad \boldsymbol{y = -3}$$

5 ABC is a triangle.
P is a point on AB such that $AP = PC = BC$
Angle $BAC = x$



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5 (a) Prove that angle $ABC = 2x$

....

(3 marks)

Use angle facts for isosceles triangles and for straight lines, justifying each statement made.
Recall that base angles of an isosceles triangle are equal, angles in a triangle add up to $180°$*, and angles on a straight line add up to* $180°$*.*

$$A\hat{C}P = C\hat{A}P = x \text{ because base angles of an isosceles triangle are equal}$$

$$A\hat{P}C = 180 - 2x \text{ because angles in a triangle add up to } 180°$$

$$B\hat{P}C = 2x \text{ because angles on a straight line add up to } 180°$$

$$A\hat{B}C = P\hat{B}C = B\hat{P}C = 2x \text{ because base angles of an isosceles triangle are equal}$$

$$\boldsymbol{Answer}: \quad (\boldsymbol{proof})$$

5 (b) You are also given that $AB = AC$

Work out the value of x.

....

....

....

$x =$ .... degrees *(3 marks)*

Use the fact that triangle ABC is isosceles to find the third angle in the triangle, form an equation, and solve for x.
Recall that base angles of an isosceles triangle are equal, and angles in a triangle add up to $180°$*.*

$$A\hat{B}C = A\hat{C}B = 2x \text{ } because\ base\ angles\ of\ an\ isosceles\ triangle\ are\ equal$$

$$A\hat{B}C + A\hat{C}B + B\hat{A}C = 2x + 2x + x = 180 \text{ } because\ angles\ in\ a\ triangle\ add\ up\ to\ 180°$$

$$5x = 180 \quad \Longrightarrow \quad x = 36°$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = 36°}$$

6 (a) Expand $3x(2x - 5y)$

Answer.. *(2 marks)*

Multiply each term in the bracket by the term outside.
Recall that $ax \times bx = abx^2$*.*

$$3x(2x - 5y) = 6x^2 - 15xy$$

$$\boldsymbol{Answer}: \quad \boldsymbol{6x^2 - 15xy}$$

6 (b) Expand and simplify $(3x+2y)(3x-4y)$

....

....

....

Answer... *(3 marks)*

Multiply each term in the first bracket by each term in the second bracket, then simplify.
Recall that, when multiplying out two brackets with two terms in each, the result – before simplifying – will have four terms altogether. Eg $(a+b)(c+d) = ac + bc + ad + bd.$

$$(3x+2y)(3x-4y) = 9x^2 + 6xy - 12xy - 8y^2 = 9x^2 - 6xy - 8y^2$$

$$\boldsymbol{Answer}\text{:} \quad \mathbf{9x^2 - 6xy - 8y^2}$$

Note: When multiplying different letters together, it is good practice to put them in alphabetical order. This makes it easier to see when terms are the same and may be combined.

6 (c) Work out the ratio $(3x+2y)(3x-4y) : 3x(2x-5y)$ when $y = 0$

Give your answer as simply as possible.

....

Answer.................... : .... *(2 marks)*

Substitute $y = 0$ into the two expressions, then cancel down as far as possible.
Recall that, when simplifying a ratio, you can multiply or divide by any number, including x.

$$When\ y = 0\text{:} \quad (3x+2y)(3x-4y) = (3x)(3x) = 9x^2 \quad and \quad 3x(2x-5y) = 3x(2x) = 6x^2$$

$$9x^2 : 6x^2 \quad \longrightarrow \quad 9:6 \quad \longrightarrow \quad 3:2$$

$$\boldsymbol{Answer}\text{:} \quad \mathbf{3:2}$$

7 $1 \leqslant m \leqslant 5$ and $-9 \leqslant n \leqslant 2$

7 (a) Work out an inequality for $m+n$.

....

....

Answer .... $\leqslant m+n \leqslant$ .... *(2 marks)*

Deal with upper and lower limits separately, and combine inequalities as you would equations. *Recall that the only difference between manipulating an equation and an inequality is that if you multiply or divide by a negative, you must reverse the inequality sign.*

$$m \leq 5 \quad and \quad n \leq 2 \quad \Rightarrow \quad m+n \leq 5+2 \quad \Rightarrow \quad m+n \leq 7$$

$$1 \leq m \quad and \quad -9 \leq n \quad \Rightarrow \quad 1 \pm 9 \leq m+n \quad \Rightarrow \quad -8 \leq m+n$$

$$\boldsymbol{Answer}: \quad \mathbf{-8 \leq m+n \leq 7}$$

7 (b) Work out an inequality for $(m+n)^2$.

....

....

Answer .... $\leqslant (m+n)^2 \leqslant$ .... *(2 marks)*

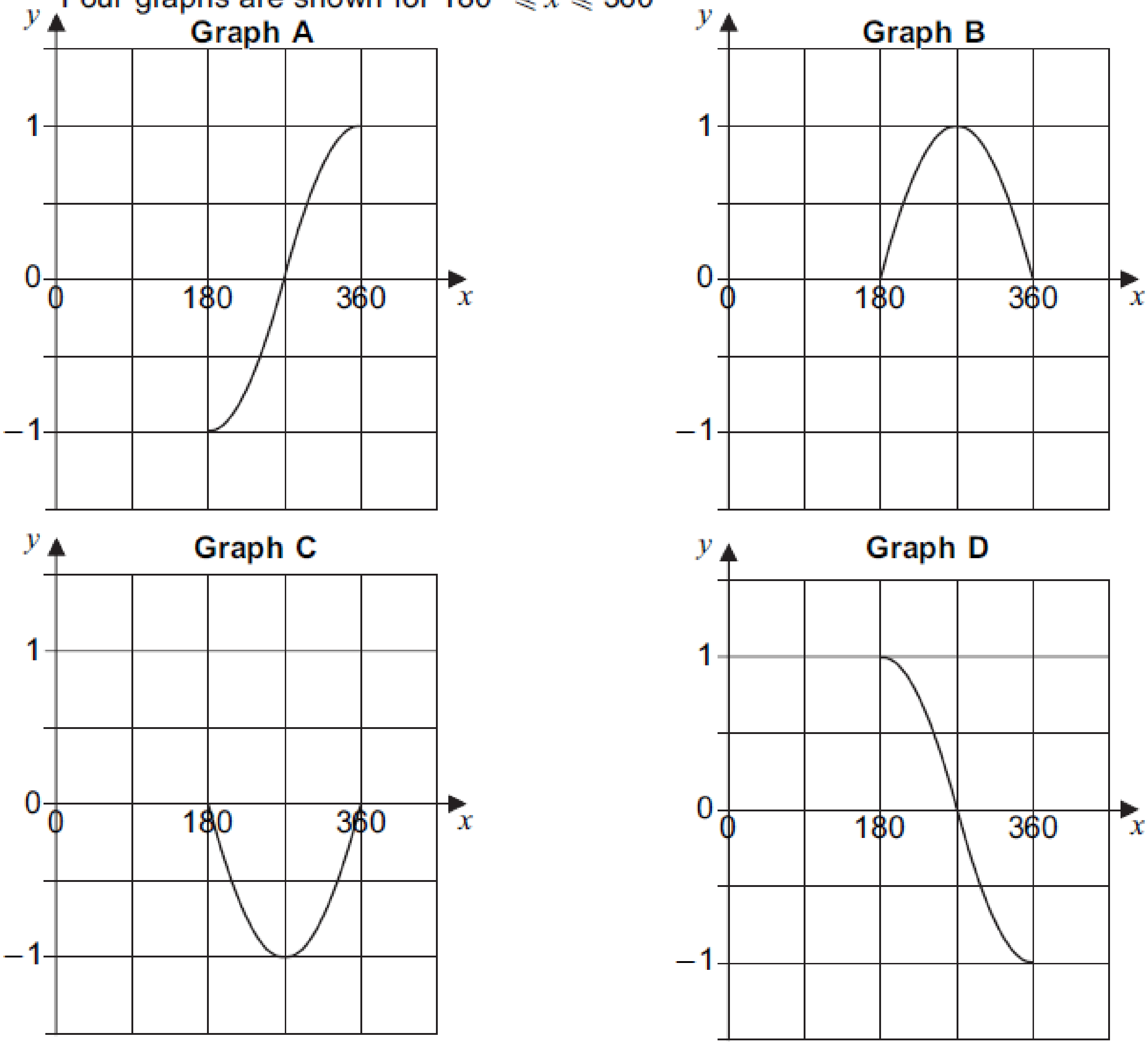
Use the limits you have calculated for $m+n$ and consider the effect of squaring.
Recall that for any real number n, $n^2 \geq 0$.

$$-8 \leq m+n \leq 7 \quad \Rightarrow \quad 0 \leq (m+n)^2 \leq 64$$

$$\boldsymbol{Answer}: \quad \mathbf{0 \leq (m+n)^2 \leq 64}$$

Note: Since $(m+n)^2 \geq 0$ for any value of m or n, it is only necessary to determine whether $(m+n)^2$ can attain this lower limit – since $m+n$ can lie anywhere between -8 and 7, it is possible to have $m+n=0$, therefore the lower limit can be reached. For the upper limit, we need the greatest possible answer to $(m+n)^2$, which occurs either at the lowest or the highest value of $m+n$. In this case, $(-8)^2 = 64$ gives us our upper limit.

8 Four graphs are shown for $180° \leqslant x \leqslant 360°$



8 (a) Which graph is $y = \sin x$?

Graph .... *(1 mark)*

Use knowledge of the trigonometric graphs to identify $y = \sin x$.

Answer*: *Graph C

Recall that the graph of $y = \sin x$ from $0°$ to $360°$ crosses the x axis at $0°$, $180°$ and $360°$, has its maximum at $90°$ and its minimum at $270°$.

8 (b) Which graph is $y = \cos x$?

Graph .... *(1 mark)*

Use knowledge of the trigonometric graphs to identify $y = \cos x$.

$$\boldsymbol{Answer}:\quad \boldsymbol{Graph\ A}$$

Recall that the graph of $y = \cos x$ from $0°$ to $360°$ crosses the x axis at $90°$ and $270°$, has its maximum at $0°$ and $360°$ and its minimum at $180°$.

9 Here is a formula.

$$5t + 3 = 4w(t + 2)$$

9 (a) Rearrange the formula to make t the subject.

....

Answer.. *(4 marks)*

Multiply out the bracket to free up the t term on the right, then move it to the left hand side, take out t as a common factor and divide.

$$5t + 3 = 4wt + 8w$$

$$5t - 4wt = 8w - 3$$

$$t(5 - 4w) = 8w - 3$$

$$t = \frac{8w - 3}{5 - 4w}$$

$$\boldsymbol{Answer}:\ \boldsymbol{t} = \frac{\mathbf{(8}\boldsymbol{w}\mathbf{ - 3)}}{\mathbf{5 - 4}\boldsymbol{w}}$$

9 (b) Work out the exact value of t when $w = -\dfrac{1}{8}$

Give your answer in its simplest form.

....

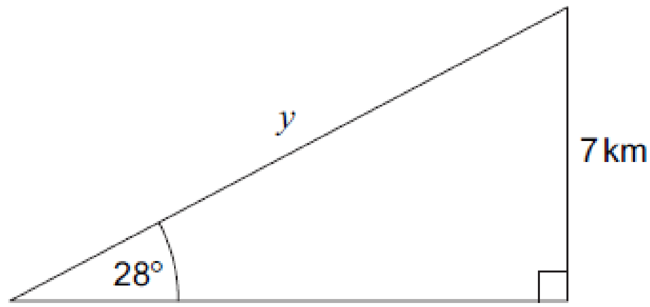
$t =$ .... *(3 marks)*

Use the rearranged form found in part a), and substitute in the value for w.
Recall that fractions of fractions can be simplified by multiplying both numerator and denominator of the whole fraction by the denominators of the fractions within.

$$t = \frac{8w - 3}{5 - 4w} = \frac{8\left(-\frac{1}{8}\right) - 3}{5 - 4\left(-\frac{1}{8}\right)} = \frac{-1 - 3}{5 + \frac{1}{2}} = -\frac{4}{\frac{11}{2}} = -\frac{8}{11}$$

$$\boldsymbol{Answer}: \quad -\frac{\mathbf{8}}{\mathbf{11}}$$

10 An aircraft flies y kilometres in a straight line at an angle of elevation of 28°.
The gain in height is 7 kilometres.



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Work out the value of y.

....

$y =$ .... km *(3 marks)*

Identify the hypotenuse, opposite and adjacent, then select the appropriate right-angled trigonometry ratio to calculate the length of the missing side.

Recall that $\sin x = \frac{opp}{hyp}$, $\cos x = \frac{adj}{hyp}$ *and* $\tan x = \frac{opp}{adj}$.

$$opp = 7km \quad hyp = y \quad \Longrightarrow \quad \sin 28 = \frac{7}{y}$$

$$\Longrightarrow \quad y = \frac{7}{\sin 28} = 14.9km\ to\ 1\ d.p.$$

$$\boldsymbol{Answer}: \quad \mathbf{14.9}\boldsymbol{km\ to\ 1\ d.p.}$$

11 A sphere has radius x centimetres.
A hemisphere has radius y centimetres.
The shapes have equal volumes.

Work out the value of $\frac{y}{x}$.

Give your answer in the form $a^{\frac{1}{3}}$ where a is an integer.

....

....

....

....

....

$\frac{y}{x} =$ .... *(3 marks)*

Use the formula for the volume of a sphere, form a formula for the volume of a hemisphere and equate the two expressions. Rearrange to find $\frac{y}{x}$ and simplify.

Recall that, for a sphere of radius r, the volume is given by $V = \frac{4}{3}\pi r^3$. Note that this formula is provided on the formula sheet at the start of the exam paper.

$$\frac{4}{3}\pi x^3 = \frac{2}{3}\pi y^3$$

$$4\pi x^3 = 2\pi y^3$$

$$2x^3 = y^3$$

$$\frac{y}{x} = \sqrt[3]{2} = 2^{\frac{1}{3}}$$

$$\boldsymbol{Answer}: \ \boldsymbol{\frac{y}{x} = 2^{\frac{1}{3}}}$$

12 Expand and simplify $(t+4)^3$

....

Answer.. *(3 marks)*

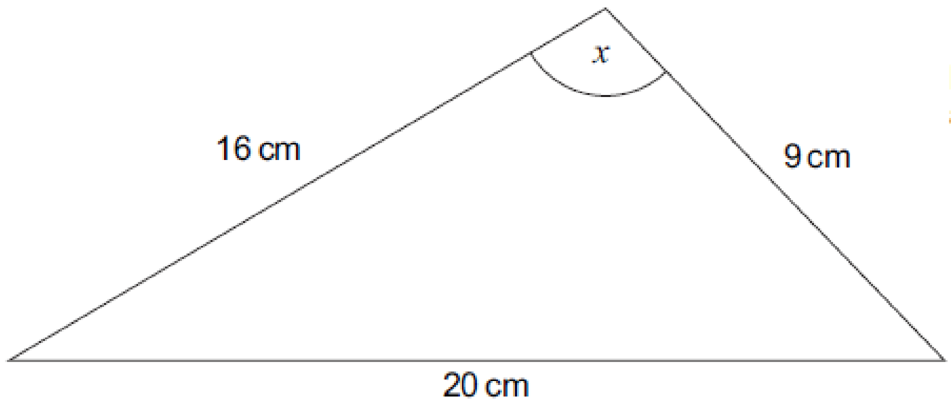
Rewrite as three brackets, then multiply one pair first to generate a quadratic. Finally, multiply each term of the quadratic by each term in the third bracket and simplify.
Recall that every single term must be multiplied by every other term, and because the linear expression is raised to the power 3 *the result will be a cubic polynomial.*

$$(t+4)^3 = (t+4)(t+4)(t+4) = (t+4)(t^2+4t+4t+16) = (t+4)(t^2+8t+16)$$

$$= t^3+8t^2+16t+4t^2+32t+64 = t^3+12t^2+48t+64$$

$$\boldsymbol{Answer}: \quad \boldsymbol{t^3+12t^2+48t+64}$$

13



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Work out angle x.

....

$x =$ .... degrees *(3 marks)*

Use the cosine rule, substitute in the lengths and solve for x.
Recall that if three sides are known, any angle can be found using $a^2 = b^2 + c^2 - 2bc\cos A$.
Note that this formula is provided on the formula sheet at the start of the exam paper.
Recall that if two sides and the angle in between are known, the third side can be calculated also using the cosine rule. If a side and the angle opposite are known, the sine rule can be used.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$20^2 = 9^2 + 16^2 - 2 \times 9 \times 16 \times \cos x$$

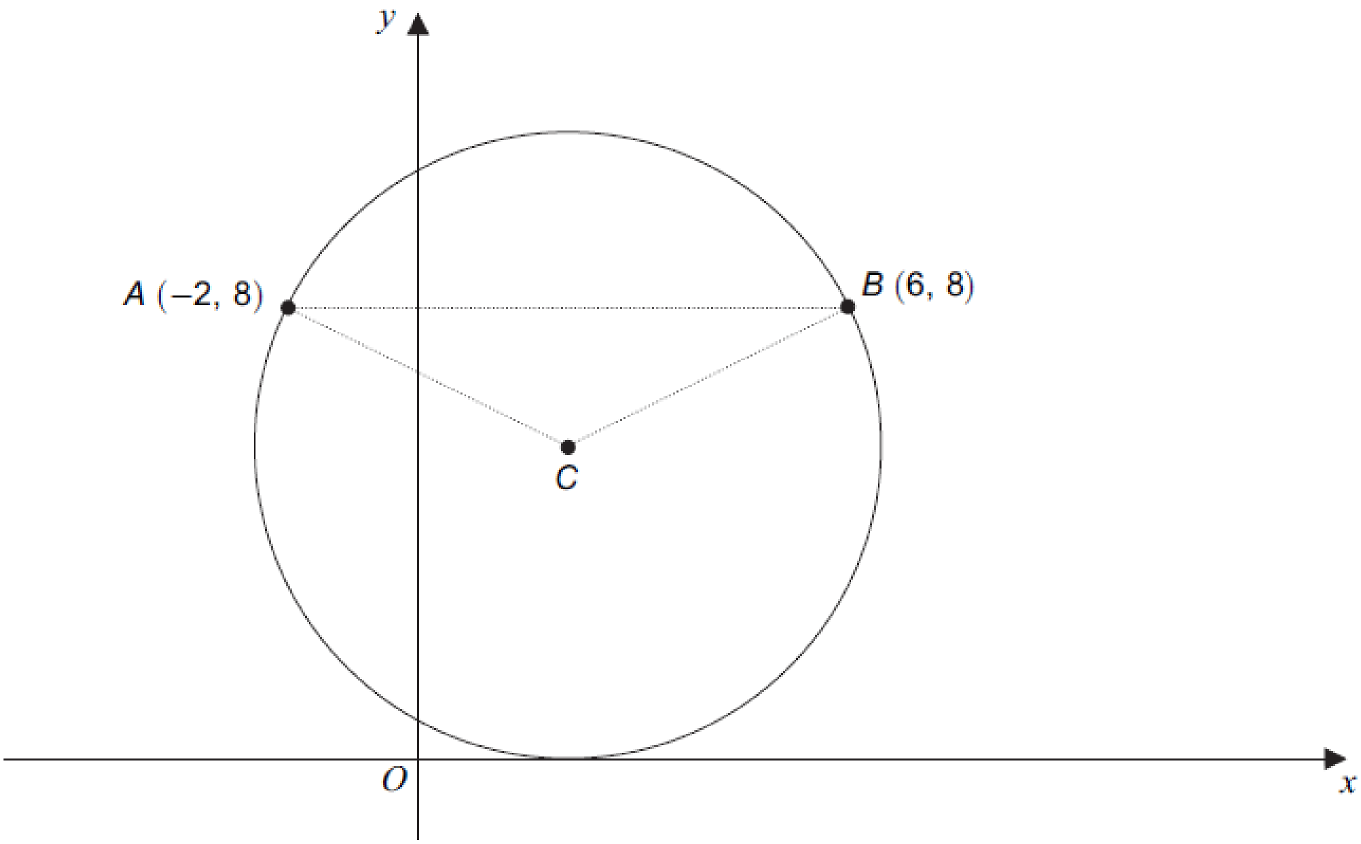
$$400 = 337 - 288\cos x$$

$$-\frac{63}{288} = \cos x$$

$$x = \cos^{-1} -\frac{63}{288} = 102.6° \ to\ 1\ d.p.$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = 102.6° \ to\ 1\ d.p.}$$

14 The sketch shows a circle, centre C, radius 5.
The circle passes through the points A $(-2, 8)$ and B $(6, 8)$.
The x-axis is a tangent to the circle.



Work out the equation of the circle.

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Answer.. *(4 marks)*

Use the symmetry of A and B to identify the x coordinate of the centre of the circle, then apply Pythagoras' theorem, using the information that the radius is 5, to find the y coordinate. Finally, substitute these values into the general formula for a circle.
Recall that a circle with radius r and centre (a, b) is described by $y = (x-a)^2 + (y-b)^2 = r^2$.

$$Midpoint\ of\ A\ and\ B\text{:}\quad \left(\frac{-2+6}{2}, \frac{8+8}{2}\right) = (2,8) \quad \Longrightarrow \quad x\ coordinate = 2$$

$$Radius = 5\ and\ x\ axis\ is\ tangent\ to\ the\ cicle \quad \Longrightarrow \quad y\ coordinate = 5$$

$$\boldsymbol{Answer}\text{:}\quad (\boldsymbol{x}-\mathbf{2})^{\mathbf{2}} + (\boldsymbol{y}-\mathbf{5})^{\mathbf{2}} = \mathbf{25}$$

15 (a) $f(x) = 3x - 5$ for all values of x.

Solve $f(x^2) = 43$

....

....

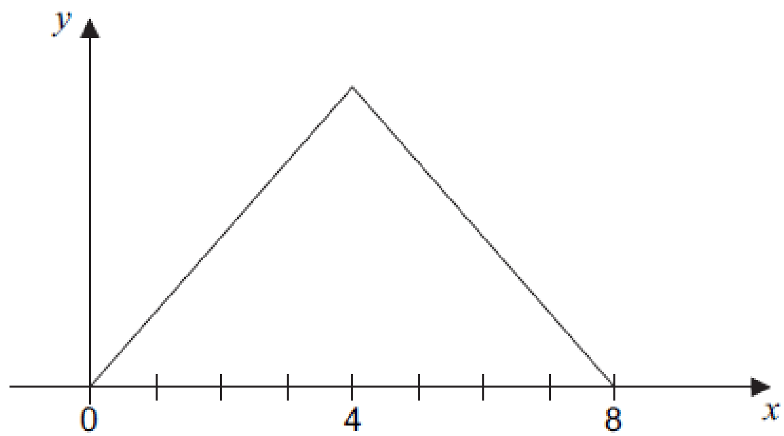
Answer.. *(4 marks)*

Substitute in x^2 to find $f(x^2)$, then equate to 43, rearrange to form a quadratic, and solve.
Recall that any quadratic equation with no x term can be factorised using the difference of two squares or solved directly by rearrangement.

$$f(x^2) = 3(x^2) - 5 \quad \Longrightarrow \quad 3x^2 - 5 = 43 \quad \Longrightarrow \quad 3x^2 = 48 \quad \Longrightarrow \quad x^2 = 16 \quad \Longrightarrow \quad x = \pm 4$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = \pm 4}$$

15 (b) A sketch of $y = g(x)$ for domain $0 \leqslant x \leqslant 8$ is shown.



The graph is symmetrical about $x = 4$
The range of $g(x)$ is $0 \leqslant g(x) \leqslant 12$

Work out the function $g(x)$.

....

$g(x) =$ ....$0 \leqslant x \leqslant 4$

....$4 < x \leqslant 8$

(5 marks)

Use the information given to determine key points of the graph, find the equation of the line from 0 to 4, then use symmetry to determine the gradient for the line from 4 to 8, and use the given coordinates to deduce the equation of the line.
Recall that a line with gradient m passing through (x_1, y_1) is given by: $y - y_1 = m(x - x_1)$.

$$First\ line\ starts\ at\ (0,0)\ and\ the\ range\ of\ g(x)\ is\ 0 \leq g(x) \leq 12 \quad \Longrightarrow \quad apex = (4,12)$$

$$\Longrightarrow \quad Gradient\ of\ first\ line: \quad m_1 = \frac{12}{4} = 3 \quad \Longrightarrow \quad Equation\ of\ first\ line: \quad y = 3x$$

$$Second\ line\ shares\ point\ (4,12)\ and\ is\ symmetrical, therefore\ has\ gradient\ m_2 = -3$$

$$\Longrightarrow \quad Equation\ of\ second\ line: \quad y - 12 = -3(x - 4) \quad or \quad y = -3x + 24$$

$$\boldsymbol{Answer}: \quad \boldsymbol{g(x)} = \begin{Bmatrix} \boldsymbol{3x} & \boldsymbol{0 \leq x \leq 4} \\ \boldsymbol{-3x + 24} & \boldsymbol{4 < x \leq 8} \end{Bmatrix}$$

16 (a) Use the factor theorem to show that $(x-1)$ and $(x-4)$ are factors of $x^3 - 21x + 20$

....

(2 marks)

Apply the factor theorem separately with each of the brackets by substituting in the appropriate value for x and showing that the result is 0.

Recall that the factor theorem states that if $(x-a)$ is a factor, a is a root, and if a is a root (that is, $f(a)=0$) then $(x-a)$ is a factor.

$$f(1) = 1 - 21 + 20 = 0 \quad \Longrightarrow \quad (x-1)\ is\ a\ factor$$

$$f(4) = 4^3 - 21(4) + 20 = 64 - 84 + 20 = 0 \quad \Longrightarrow \quad (x-4)\ is\ a\ factor$$

$$\boldsymbol{Answer}: \ (\boldsymbol{proof})$$

16 (b) Show that $(x-1)$ and $(x-4)$ are also factors of $x^3 - 10x^2 + 29x - 20$

....

(2 marks)

Apply the factor theorem as above, making sure to demonstrate clearly that the result is 0.

$$f(1) = 1^3 - 10(1^2) + 29(1) - 20 = 1 - 10 + 29 - 20 = 0 \quad \Longrightarrow \quad (x-1)\ is\ a\ factor$$

$$f(4) = 4^3 - 10(4^2) + 29(4) - 20 = 64 - 160 + 116 - 20 = 0 \quad \Longrightarrow \quad (x-4)\ is\ a\ factor$$

$$\boldsymbol{Answer}: \ (\boldsymbol{proof})$$

16 (c) Hence, simplify fully $\dfrac{x^3 - 21x + 20}{x^3 - 10x^2 + 29x - 20}$

....

Answer.. *(3 marks)*

Use the factors found in parts a) and b) to fully factorise the numerator and denominator, then simplify by dividing top and bottom by any common factors.
Recall that additional factors can be found by using inspection (see below).

$$x^3 - 21x + 20 = (x-1)(x-4)(\ldots) = (x^2 - 5x + 4)(\ldots)$$

$$To\ produce\ an\ x^3\ term\ we\ need\ x\ in\ the\ final\ bracket, and\ to\ produce\ 20\ we\ need\ 5$$

$$x^3 - 21x + 20 = (x^2 - 5x + 4)(x+5)$$
$$Note: we\ can\ verify\ this\ by\ multiplying\ out\ the\ brackets\ if\ necessary$$

$$x^3 - 10x^2 + 29x - 20 = (x-1)(x-4)(\ldots) = (x^2 - 5x + 4)(\ldots)$$

$$To\ produce\ an\ x^3\ term\ we\ need\ x\ in\ the\ final\ braket, and\ to\ produce - 20\ we\ need - 5$$

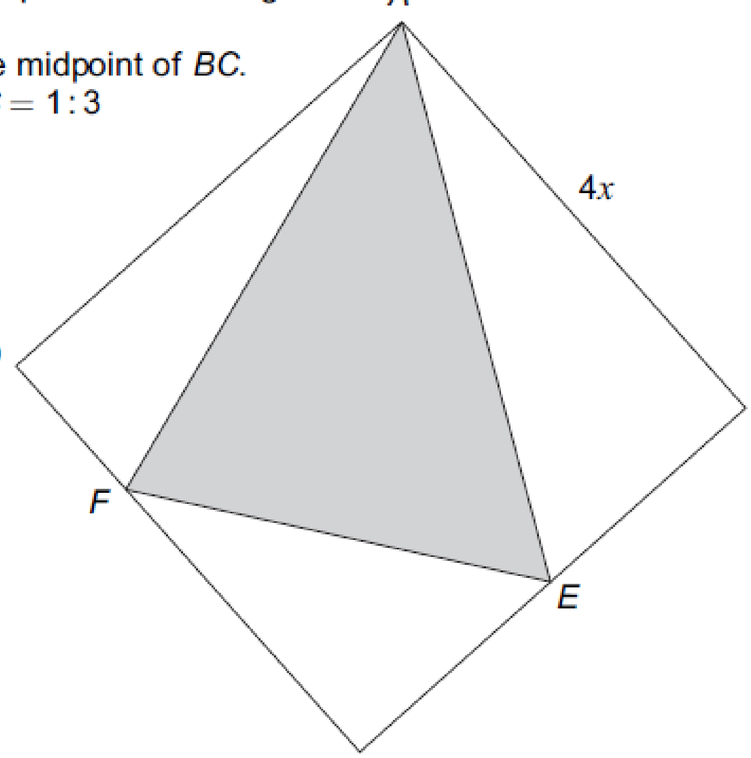
$$x^3 - 10x^2 + 29x + 20 = (x^2 - 5x + 4)(x-5)$$

$$\frac{x^3 - 21x + 20}{x^3 - 10x^2 + 29x + 20} = \frac{(x-1)(x-4)(x+5)}{(x-1)(x-4)(x-5)} = \frac{x+5}{x-5}$$

$$\boldsymbol{Answer}: \ \ \boldsymbol{\frac{x+5}{x-5}}$$

17 *ABCD* is a square of side length $4x$.

E is the midpoint of *BC*.
$DF : FC = 1 : 3$



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You are given that

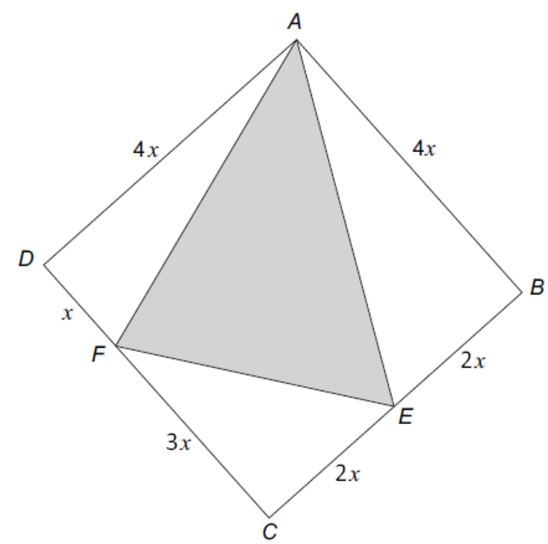
$$\text{area of triangle } AEF = kx^2$$

Work out the value of k.

....

$k =$ .... *(5 marks)*

Use the information given to more fully label the diagram, then find the area of the three non-shaded triangles. Subtract from the total for the square to find the shaded area and hence k.



$$Area\ of\ ADF + ABE + ECF = 2x^2 + 4x^2 + 3x^2 = 9x^2$$

$$Area\ of\ ABCD = 16x^2$$

$$Area\ of\ AEF = 16x^2 - 9x^2 = 7x^2$$

$$\boldsymbol{Answer}: \quad \boldsymbol{k = 7}$$

18 $(x-5)^2 + a \equiv x^2 + bx + 28$

Work out the values of a and b.

....

....

....

....

$a =$ .... $b =$ .... *(3 marks)*

Multiply out the left hand side and compare coefficients of x and the constants.
Recall that, since the left hand side is in completed square form, some careful inspection would be sufficient to determine the value of b *and hence the value of* a*.*

$$(x-5)^2 + a = x^2 - 10x + 25 + a \quad \Longrightarrow \quad x^2 - 10x + 25 + a \equiv x^2 + bx + 28$$

$$\Longrightarrow \quad -10 = b \quad and \quad 25 + a = 28 \quad so \quad a = 3$$

$$\boldsymbol{Answer}: \quad \boldsymbol{a = 3} \quad \boldsymbol{b = -10}$$

19 Solve the simultaneous equations

$$x + y = 4$$
$$y^2 = 4x + 5$$

Do **not** use trial and improvement.

....

Answer... *(6 marks)*

Rearrange the linear equation and substitute x into the quadratic equation. Solve this quadratic equation, then substitute back into the linear equation to find the other unknown.
Recall that a quadratic can be solved either by factorising (if possible), completing the square or applying the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x + y = 4 \quad \Rightarrow \quad x = 4 - y$$

$$y^2 = 4x + 5 \quad \Rightarrow \quad y^2 = 4(4 - y) + 5 = 16 - 4y + 5 = 21 - 4y$$

$$\Rightarrow \quad y^2 + 4y - 21 = 0 \quad \Rightarrow \quad (y + 7)(y - 3) = 0 \quad \Rightarrow \quad y = -7 \quad or \quad y = 3$$

$$Substituting\ into\ x + y = 4 \quad \Rightarrow \quad for\ y = -7 \quad x = 11 \quad and\ for\ y = 3 \quad x = 1$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = 1, y = 3 \quad and \quad x = 11, y = -7}$$

20 For what values of x is $y = 150x - 2x^3$ an increasing function?

....

Answer.. *(4 marks)*

Differentiate the function and determine when this expression is positive.
Recall that a function is increasing whenever $\frac{dy}{dx} > 0$. *Recall that to solve a quadratic inequality it is necessary to determine critical points and consider the regions either side and between.*

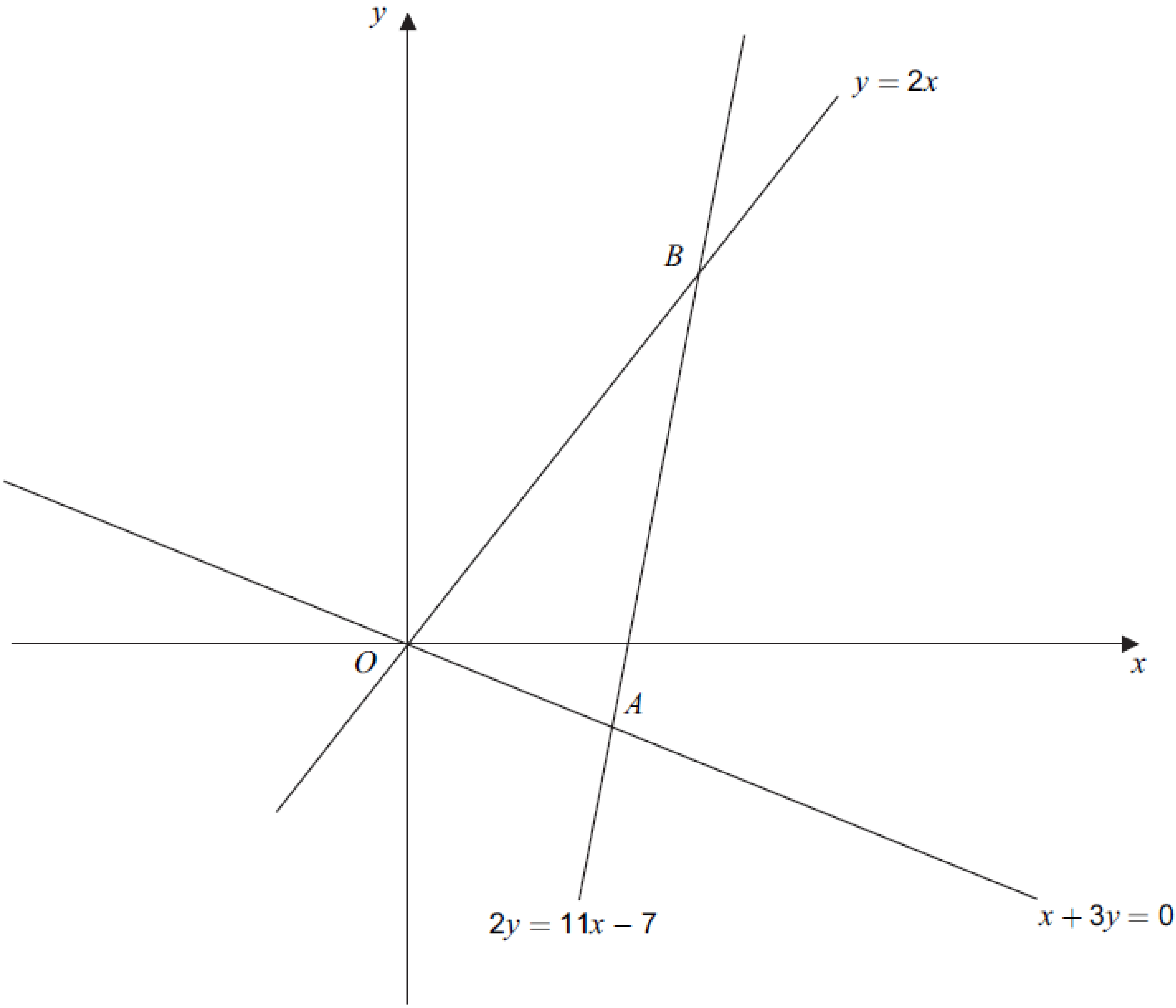
$$\frac{dy}{dx} = 150 - 6x^2 > 0 \quad \Rightarrow \quad 25 - x^2 > 0 \quad \Rightarrow \quad x^2 < 25 \quad \Rightarrow \quad -5 < x < 5$$

$$\boldsymbol{Answer}: \quad \boldsymbol{-5 < x < 5}$$

21 The equations of three straight lines are

$$y = 2x \qquad x + 3y = 0 \qquad 2y = 11x - 7$$

The lines intersect at the points O, A and B as shown on this sketch.



Show that length OB = length AB

....

(6 marks)

Solve $2y = 11x - 7$ and $x + 3y = 0$ simultaneously to find the coordinates of A, then solve $2y = 11x - 7$ and $y = 2x$ simultaneously to find the coordinates of B. Finally, use Pythagoras' theorem to determine the two lengths in question.

Recall that the distance between (x_1, y_1) and (x_2, y_2) is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$x + 3y = 0 \quad \Longrightarrow \quad x = -3y$$

$$2y = 11x - 7 \quad \Longrightarrow \quad 2y = 11(-3y) - 7 \quad \Longrightarrow \quad 2y = -33y - 7 \quad \Longrightarrow \quad 35y = -7$$

$$\Longrightarrow \quad y = -\frac{1}{5} \quad \Longrightarrow \quad x = -3\left(-\frac{1}{5}\right) = \frac{3}{5} \quad \Longrightarrow \quad Coordinates\ of\ A: \quad \left(\frac{3}{5}, -\frac{1}{5}\right)$$

$$y = 2x \quad and \quad 2y = 11x - 7 \quad \Longrightarrow \quad 2(2x) = 11x - 7 \quad \Longrightarrow \quad 7 = 7x$$

$$\Longrightarrow \quad x = 1 \quad \Longrightarrow \quad y = 2 \quad \Longrightarrow \quad Coordinates\ of\ B: \quad (1,2)$$

$$OB = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$AB = \sqrt{\left(1 - \frac{3}{5}\right)^2 + \left(2 + \frac{1}{5}\right)^2} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{11}{5}\right)^2} = \sqrt{\frac{125}{25}} = \sqrt{5}$$

$$Therefore\ OB = AB$$

$$\boldsymbol{Answer}: \quad (\boldsymbol{proof})$$

22 The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps point P to point Q.

The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R.

Point R is $(-4, 3)$.

Work out the coordinates of point P.

Answer (.........................., ...) *(5 marks)*

Apply the first matrix followed by the second matrix, to a point P using (x, y), and hence find the point R in terms of x and y. Solve the resulting equations to find x and y.

Recall that, to apply a 2 by 2 matrix to a point (or to multiply a 2 by 2 vector by a 2 by 1 vector), follow the rule $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

$$Q = \begin{bmatrix} -y \\ -x \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -y \\ -x \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$R = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \Rightarrow \quad y = 4 \quad and \quad x = 3$$

$$\boldsymbol{Answer}: \quad (\mathbf{3}, \mathbf{4})$$

Alternative method – combining the transformation matrices first:

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \Rightarrow \quad y = 4 \quad and \quad x = 3$$

$$\boldsymbol{Answer}: \quad (\mathbf{3}, \mathbf{4})$$

Recall that two 2 by 2 matrices can be multiplied using the following rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

23 The curve $y = f(x)$ is such that $\frac{dy}{dx} = -x(x-2)^2$

The stationary points of the curve are at $\left(0, \frac{4}{3}\right)$ and $(2, 0)$.

Determine the nature of each stationary point.
You **must** show your working.

....

(4 marks)

Expand the brackets, differentiate a second time and substitute in the values of x for the two points. Use the values of $\frac{d^2y}{dx^2}$ to determine the nature of the stationary points.

Recall that at a minimum $\frac{d^2y}{dx^2} > 0$, *at a maximum* $\frac{d^2y}{dx^2} < 0$ *and at a point of inflection* $\frac{d^2y}{dx^2} = 0$, *although this is not sufficient to prove a point of inflection.*

$$\frac{dy}{dx} = -x(x-2)^2 = -x(x^2 - 4x + 4) = -x^3 + 4x^2 - 4x$$

$$\frac{d^2y}{dx^2} = -3x^2 + 8x - 4 \quad At\ x = 0 \quad \frac{d^2y}{dx^2} = -4 < 0 \quad \Rightarrow \quad Maximum$$

$$At\ x = 2 \quad \frac{d^2y}{dx^2} = -3(2^2) + 8(2) - 4 = 0$$

$$At\ x = 1 \quad \frac{dy}{dx} = -1(1-2)^2 = -1 \quad At\ x = 3 \quad \frac{dy}{dx} = -3(3-2)^2 = -3$$

$$\frac{d^2y}{dx^2} = 0 \quad and \quad \frac{dy}{dx} < 0 \quad either\ side \quad \Rightarrow \quad Point\ of\ Inflection$$

$$\boldsymbol{Answer}: \quad \boldsymbol{maximum\ at}\ \left(\mathbf{0}, \frac{\mathbf{4}}{\mathbf{3}}\right) \quad \boldsymbol{and} \quad \boldsymbol{point\ of\ inflection\ at}\ (\mathbf{2}, \mathbf{0})$$

Note: provided the gradient is the same either side of a stationary point, it will be a point of inflection.

Opposite is the graph of the function in question – note the maximum at $\left(0, \frac{4}{3}\right)$ *at the point of inflection at* $(0,2)$.