Further Maths Level 2 Exam Paper Solutions (June 2012)

Paper 1 – Non-Calculator

1 $\mathrm{f}(x) = 2x^2 + 7$ for all values of x.

1 (a) What is the value of $\mathrm{f}(-1)$?

Answer... *(1 mark)*

Substitute the value in brackets as x into the function, and evaluate:
Recall that $(-1)^2 = 1$.

$$f(-1) = 2(-1)^2 + 7 = 2 \times 1 + 7 = 9$$

$$\boldsymbol{Answer}: \quad \mathbf{9}$$

1 (b) What is the range of $\mathrm{f}(x)$?

Answer... *(1 mark)*

Determine what value the function might take given that x can be any value.
Recall that the range of a function is the set of all possible output values.
Recall that $x^2 \geq 0$ *for all values of* x.

$$x^2 \geq 0 \quad \Longrightarrow \quad 2x^2 \geq 0 \quad \Longrightarrow \quad 2x^2 + 7 \geq 7$$

$$\boldsymbol{Answer}: \quad \boldsymbol{f(x) \geq 7}$$

Note: Range is always given in terms of the function, $f(x)$, as it is the range of output values. Domain would be given in terms of the variable x, since it is to do with input values.

2 $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Work out the matrix **AB**.

AB = ... *(2 marks)*

Write the matrices next to each other and follow the technique for matrix multiplication.
Recall that the order for matrix multiplication is important.

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \times 5 + 0 \times 4 \\ 1 \times 5 + 3 \times 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \end{pmatrix}$$

$$\boldsymbol{Answer}: \quad \begin{pmatrix} \mathbf{10} \\ \mathbf{17} \end{pmatrix}$$

Note: When multiplying a 2×2 *matrix by a* 2×1 *matrix, the result will be a* 2×1 *matrix.*
When multiplying a 2×2 *matrix by a* 2×2 *matrix, the result will be a* 2×2 *matrix.*

3 Work out the greatest integer value of x that satisfies the inequality $3x + 10 < 1$

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Answer.. *(2 marks)*

Solve the inequality as you would a linear equation, by rearranging to get x alone.
Recall that if you multiply or divide by a negative number, you must reverse the inequality.

$$3x + 10 < 1 \quad \Rightarrow \quad 3x < -9 \quad \Rightarrow \quad x < -\frac{9}{3} \quad \Rightarrow \quad x < -3$$

$$Since\ x\ cannot\ be\ equal\ to -3\ the\ greatest\ integer\ value\ must\ be -4$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = -4}$$

4 (a) Factorise fully $2x^2 - 2x - 40$

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Answer.. *(3 marks)*

First, notice that all terms have a common factor of 2 which can be taken out.
Then write the quadratic expression in two brackets, each containing an x term and a number.
Recall that the common factor of 2 must form part of the final answer.
Recall that, since the constant term is negative, the signs in the brackets must be different.

$$2x^2 - 2x - 40 = 2(x^2 - x - 20) = 2(x \pm \cdots)(x \mp \cdots)$$

$$Possible\ factor\ pairs: \quad (1,20) \quad (2,10) \quad (4,5)$$

$$Since\ the\ pair\ chosen\ must\ have\ a\ difference\ of\ 1:$$

$$= 2(x+4)(x-5)$$

$$\boldsymbol{Answer}: \quad \mathbf{2(}\boldsymbol{x}\mathbf{+4)(}\boldsymbol{x}\mathbf{-5)}$$

Note: If this were a quadratic equation $2x^2 - 2x - 40 = 0$ we could divide through by 2 and solve by writing as $(x+4)(x-5) = 0$. However, since it is an expression, and must remain simply a rearrangement of the original form, the factor of 2 must remain part of the solution.

4 (b) Factorise fully $(x + y)^2 + (x + y)(2x + 5y)$

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Answer.. *(3 marks)*

Take out a common factor of $(x + y)$ at the start, then simplify by combining like terms.
Recall that if factors can be removed at the start, it will make the process of fully simplifying much more straightforward than multiplying everything out before looking for factors.

$$(x+y)^2 + (x+y)(2x+5y) = (x+y)[(x+y)+(2x+5y)] = (x+y)(3x+6y)$$

$$= 3(x+y)(x+2y)$$

$$\boldsymbol{Answer}: \quad \mathbf{3}(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}+\mathbf{2}\boldsymbol{y})$$

Note: It is important to check at the end that no further factors can be taken out.
If you do multiply everything out at the start, it is still possible to factorise, but you should remember that $(x+y)^2 = x^2 + 2xy + y^2$, not $x^2 + y^2$.

Alternative method – not taking out the factor of $(x + y)$:

$$(x+y)^2 + (x+y)(2x+5y) = (x^2 + 2xy + y^2) + (2x^2 + 5xy + 2xy + 5y^2)$$

$$= 3x^2 + 9xy + 6y^2 = 3(x^2 + 3xy + 2y^2) = 3(x+y)(x+2y)$$

$$\boldsymbol{Answer}: \quad \mathbf{3}(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}+\mathbf{2}\boldsymbol{y})$$

Note: This final step does require more thought than standard factorising of quadratics, which is why identifying and dealing with common factors at the beginning (as shown in the first method) should be the preferred approach.

5 Simplify $(2cd^4)^3$

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Answer.. *(2 marks)*

Since all the terms in the bracket are multiplied, it is sufficient to raise each one individually to the power of 3.

Recall that $(xy)^n = x^n y^n$ *and that* $(x^n)^m = x^{nm}$.

$$(2cd^4)^3 = 2^3c^3d^{12} = 8c^3d^{12}$$

$$\boldsymbol{Answer}: \quad \mathbf{8}\boldsymbol{c^3d^{12}}$$

Note: Remember that $2^3 = 2 \times 2 \times 2 = 8$*, not* $2 + 2 + 2 = 6$.

6 Solve the simultaneous equations

$$2y = 3x + 4$$

$$2x = -3y - 7$$

Do **not** use trial and improvement.

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Answer.. *(4 marks)*

Multiply each equation through by a number which will allow them to be combined using the elimination method, cancelling out either x or y.
Recall that once equations have the same (or the exact negative) coefficient for one of the variables, they can either be rearranged and then effectively be added or subtracted from one another, or a sensible substitution can be made, to eliminate this variable.

$$Equation\ 1: \quad 2y = 3x + 4 \quad \Longrightarrow \quad 4y = 6x + 8$$

$$Equation\ 2: \quad 2x = -3y - 7 \quad \Longrightarrow \quad 6x = -9y - 21$$

$$Rearranging\ equation\ 1: \quad 6x = 4y - 8$$

$$Since\ 6x = -9y - 21 \quad and \quad 6x = 4y - 8: \quad -9y - 21 = 4y - 8$$

$$\Longrightarrow \quad -13 = 13y \quad \Longrightarrow \quad y = -1$$

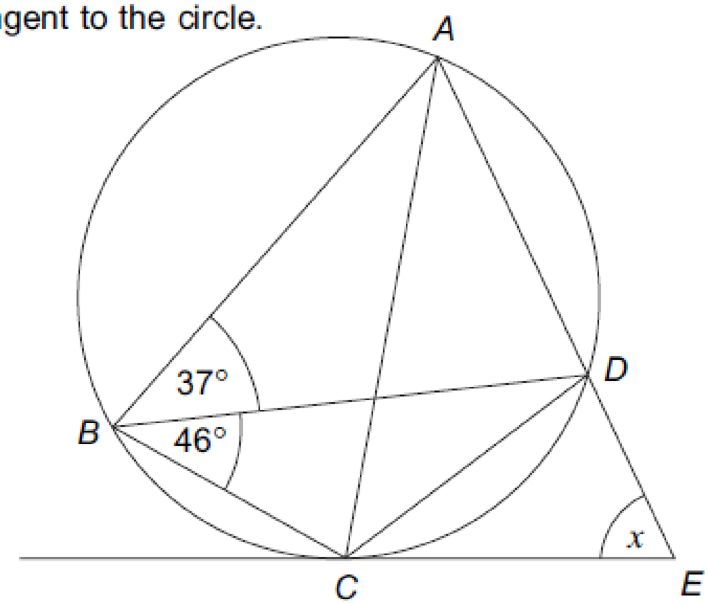
$$Substituting\ back\ into\ equation\ 1: \quad 2(-1) = 3x + 4 \quad \Longrightarrow \quad -6 = 3x \quad \Longrightarrow \quad x = -2$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = -2} \quad \boldsymbol{y = -1}$$

Note: In this form, any method will require some manipulation of the equations, but there are a number of equally valid approaches.

7 The diagram shows a cyclic quadrilateral *ABCD*.

ADE is a straight line.
CE is a tangent to the circle.



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Work out the size of angle x.

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$x =$ .... degrees *(3 marks)*

Apply the circle theorems to unknown angles until there is enough information to find x.
Recall that angles in the same segment are equal, and that the angle a tangent makes with a chord is equal to the angle made by the chord in the alternate segment.

$$\textit{Since angles in the same segment are equal:}$$
$$C\hat{A}D = C\hat{B}D = 46° \quad \textit{and} \quad A\hat{C}D = A\hat{B}D = 37°$$

$$\textit{Since angles in a triangle add up to } 180°: \quad A\hat{D}C = 180 - (37 + 46) = 97°$$

$$\textit{Since angles on a straight line add up to } 180°: \quad E\hat{D}C = 180 - 97 = 83°$$

$$\textit{By the alternate segment theorem:} \quad D\hat{C}E = D\hat{B}C = 46°$$

$$\textit{Since angles in a triangle add up to } 180°: \quad D\hat{E}C = 180 - (83 + 46) = 51°$$

$$\boldsymbol{Answer}: \quad \boldsymbol{x = 51}°$$

8 A curve has equation $y = x^3 + 5x^2 + 1$

8 (a) When $x = -1$, show that the value of $\frac{dy}{dx}$ is -7.

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(2 marks)

Use the correct differentiation method for powers of x, and substitute in the value $x = -1$.
Recall that if $y = ax^n$ *then* $\frac{dy}{dx} = anx^{n-1}$ *and that the differential of a constant is always* 0.

$$y = x^3 + 5x^2 + 1 \quad \Longrightarrow \quad \frac{dy}{dx} = 3x^2 + 10x$$

$$At\ x = -1: \ \frac{dy}{dx} = 3(-1)^2 + 10(-1) = 3 - 10 = -7$$

$$\boldsymbol{Answer}: \ \boldsymbol{\frac{dy}{dx} = -7\ as\ required}$$

8 (b) Work out the equation of the tangent to the curve $y = x^3 + 5x^2 + 1$ at the point where $x = -1$

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Answer.. *(4 marks)*

Use the previous answer to give the gradient, and substitute $x = -1$ into the original function to find a corresponding y coordinate. Then use the straight line equation formula.
Recall that a line with gradient m passing through (x_1, y_1) is given by: $y - y_1 = m(x - x_1)$.

$$At\ x = -1: \quad y = (-1)^3 + 5(-1)^2 + 1 = -1 + 5 + 1 = 5$$

$$m = -7 \quad and \quad (x_1, y_1) = (-1,5) \quad \Longrightarrow \quad y - 5 = -7(x + 1)$$

$$\boldsymbol{Answer}: \quad \boldsymbol{y - 5 = -7(x + 1)}$$

Note: This equation could be rearranged to give a more usual form such as $y = -7x - 2$ or even $y + 7x + 2 = 0$, but the form used above is sufficient.

9 Write this ratio in its simplest form

$$\sqrt{12} \ : \ \sqrt{48} \ : \ \sqrt{300}$$

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Answer.......................... : .... : .... *(3 marks)*

Fully simplify each surd by finding square factors, then multiply or divide each part by the same thing to write the entire ratio, if possible, in terms of whole numbers.
Recall that $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

$$\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3} \quad and \quad \sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3} \quad and \quad \sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3}$$

$$2\sqrt{3} : 4\sqrt{3} : 10\sqrt{3} \quad \longrightarrow \quad 2:4:10 \quad \longrightarrow \quad 1:2:5$$

$$\boldsymbol{Answer}: \quad \mathbf{1:2:5}$$

10 The n^{th} term of the linear sequence 2 7 12 17 ... is $5n - 3$

A new sequence is formed by squaring each term of the linear sequence and adding 1 .

Prove algebraically that **all** the terms in the new sequence are multiples of 5 .

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(4 marks)

Use the n^{th} term rule given, apply the changes described and rearrange the resulting expression to demonstrate that it must always have 5 as a factor.

$$(5n-3)^2+1=(25n^2-30n+9)+1=25n^2-30n+10$$

$$=5(5n^2-6n+2)$$

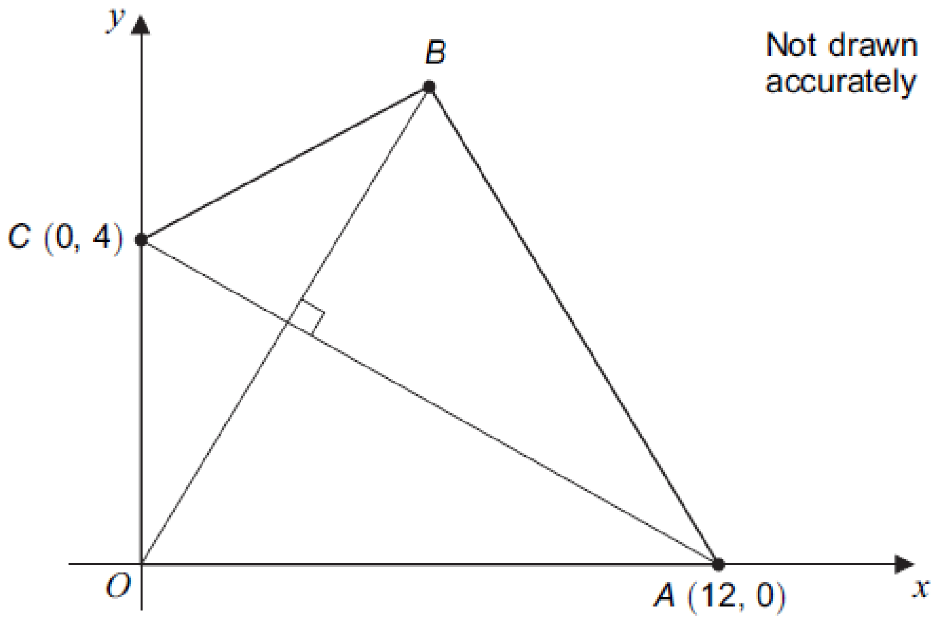
$$5n^2-6n+2 \textit{ gives an integer answer for all } n \quad \Longrightarrow \quad 5(5n^2-6n+2) \textit{ is a multiple of } 5$$

$$\boldsymbol{Answer}: \quad (\boldsymbol{proof})$$

Note: It is particularly important with proof questions that each part of the method is clear, as well as any logical steps which have been made. State your reasoning at each stage.

Note: It would also be sufficient to prove the result if the quadratic sequence described were written out and the n^{th} term generated from this using a clear and valid method. However, the algebraic proof shown above is the quickest and simplest way of demonstrating the required result.

11 *OABC* is a kite.



11 (a) Work out the equation of *AC*.

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Answer .... *(2 marks)*

Find the gradient between the two points, and use the y intercept at C to generate an equation of the form $y = mx + c$.

Recall that the gradient between the two points (x_1, y_1) *and* (x_2, y_2) *is given by* $\frac{y_2 - y_1}{x_2 - x_1}$. *This can most easily be remembered conceptually, as* $\frac{y\text{-}step}{x\text{-}step}$.

$$c = 4 \quad and \quad m = \frac{0-4}{12-0} = -\frac{4}{12} = -\frac{1}{3}$$

$$y = mx + c \quad \Longrightarrow \quad y = -\frac{1}{3}x + 4$$

$$\boldsymbol{Answer}: \quad \boldsymbol{y = -\frac{1}{3}x + 4}$$

Note: This equation could be given in any valid form, eg $3y + x - 12 = 0$.

11 (b) Work out the coordinates of B.

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Answer (......................... , ...) *(6 marks)*

Use the fact that OB is perpendicular to AC to find the gradient, and use the y intercept at O to generate an equation of the form $y = mx + c$. Use the fact that the crossing point of the lines OB and AC is the midpoint of the line segment OB to find the coordinates of B.
Recall that perpendicular lines with gradients m_1 and m_2 must satisfy the condition $m_1 m_2 = -1$.
To find the crossing point of two lines, solve their equations simultaneously.

$$Gradient\ of\ AC = -\frac{1}{3} \quad \Longrightarrow \quad Gradient\ of\ OB = 3 \quad and \quad c = 0 \quad \Longrightarrow \quad y = 3x$$

$$AC\ crosses\ OB\ when: \quad y = 3x \quad and \quad y = -\frac{1}{3}x + 4$$

$$\Longrightarrow \quad 3x = -\frac{1}{3}x + 4 \quad \Longrightarrow \quad 9x = -x + 12 \quad \Longrightarrow \quad 10x = 12 \quad \Longrightarrow \quad x = 1.2$$

$$Substituting\ back\ into\ y = 3x: \quad y = 3(1.2) = 3.6$$

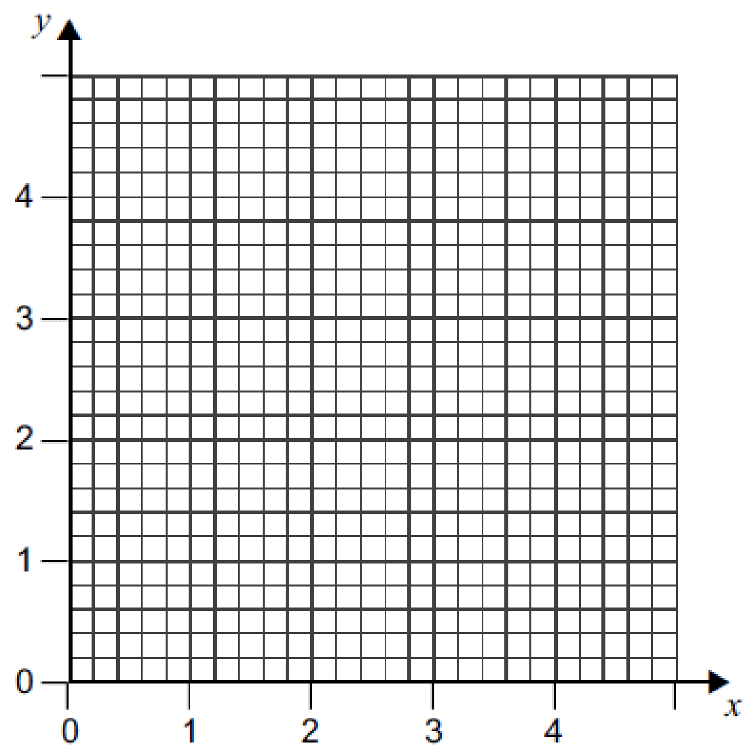
$$(1.2,3.6)\ is\ the\ midpoint\ of\ the\ line\ OB \quad \Longrightarrow \quad B\ has\ coordinates\ (2.4,7.2)$$

$$\boldsymbol{Answer}: \quad (\mathbf{2.4, 7.2})$$

12 (a) A graph passes through $(0, 0)$.

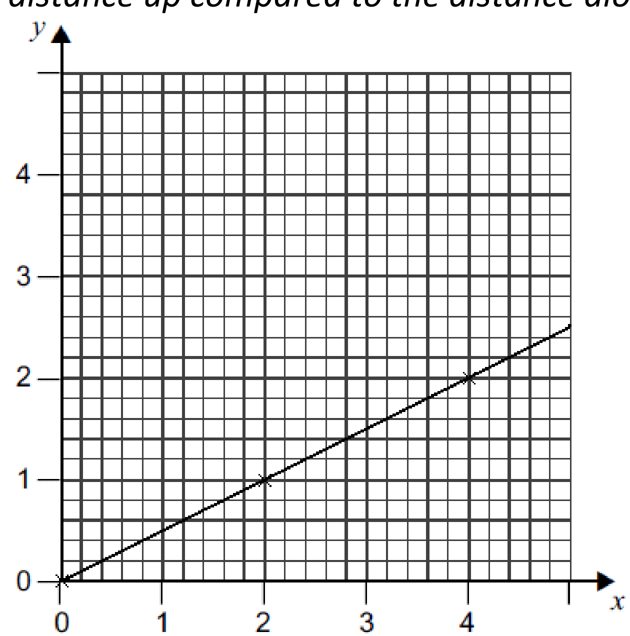
The rate of change of y with respect to x is always $\frac{1}{2}$.

Draw the graph of y for values of x from 0 to 4 .



(1 mark)

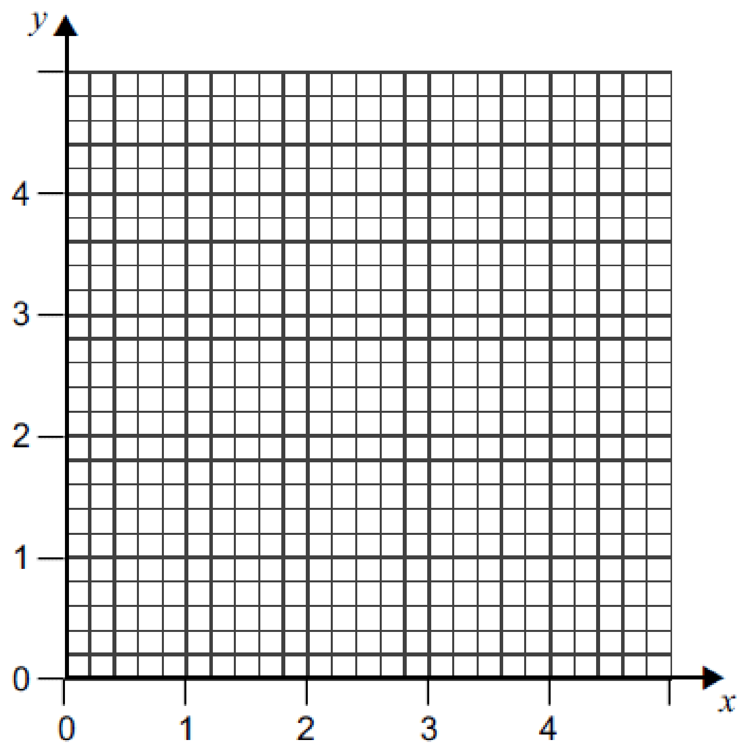
Use the starting point, and the given gradient to construct the graph.
Recall that if a gradient is a constant the graph will be a straight line.
The gradient represents the distance up compared to the distance along.



12 (b) A graph passes through $(1, 2)$.

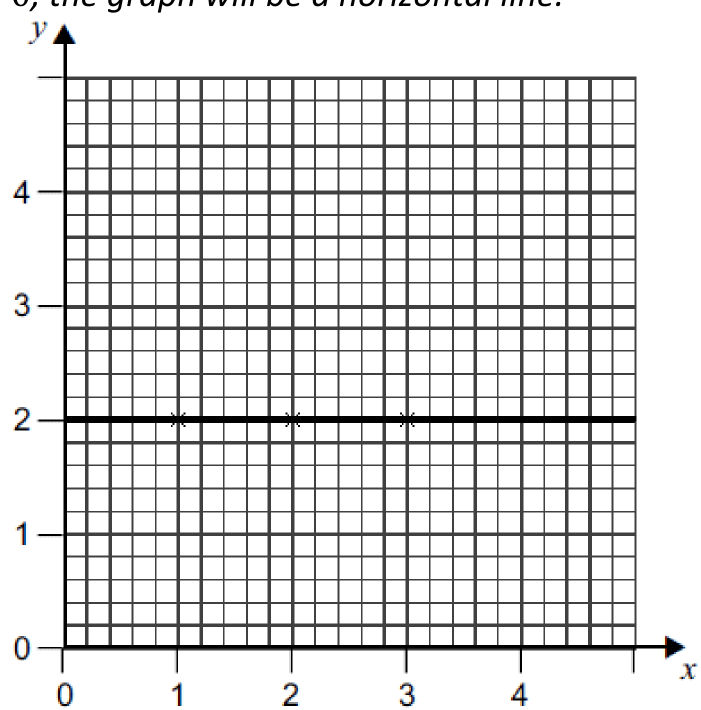
The rate of change of y with respect to x is always 0.

Draw the graph of y for values of x from 0 to 4.



(1 mark)

Use the starting point, and the given gradient to construct the graph.
Recall that if a gradient is 0*, the graph will be a horizontal line.*



12 (c) $y = 2x^3 + ax$, where a is a constant.

The value of $\frac{dy}{dx}$ when $x = 2$ is twice the value of $\frac{dy}{dx}$ when $x = -1$

Work out the value of a.

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$a =$ .... *(5 marks)*

Differentiate the expression and generate an expression by substituting in the value $x = 2$. Do the same for $x = -1$ and then use the information given to construct an equation. Solve for a. *Recall that the differential of ax^n is ax^{n-1}, so the differential of ax is a.*

$$y = 2x^3 + ax \quad \Longrightarrow \quad \frac{dy}{dx} = 6x^2 + a$$

$$At\ x = 2:\ \ \frac{dy}{dx} = 6(2)^2 + a = 24 + a$$

$$At\ x = -1:\ \ \frac{dy}{dx} = 6(-1)^2 + a = 6 + a$$

$$24 + a = 2(6 + a) \quad \Longrightarrow \quad 24 + a = 12 + 2a \quad \Longrightarrow \quad 12 = a$$

$$\boldsymbol{Answer}: \quad \boldsymbol{a = 12}$$

13 Simplify $\dfrac{x^2+4x-12}{x^2-25} \div \dfrac{x+6}{x^2-5x}$

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Answer.. *(5 marks)*

Rewrite as a multiplication in order to combine the rational expressions (fractions), and factorise expressions where possible. Finally, simplify by cancelling out common factors.
Recall that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

$$\frac{x^2+4x-12}{x^2-25} \div \frac{x+6}{x^2+5x} = \frac{x^2+4x-12}{x^2-25} \times \frac{x^2+5x}{x+6} = \frac{(x+6)(x-2)}{(x+5)(x-5)} \times \frac{x(x+5)}{x+6}$$

$$= \frac{(x+6)(x-2)x(x+5)}{(x+5)(x-5)(x+6)} = \frac{x(x-2)}{x-5}$$

$$\boldsymbol{Answer}: \ \boldsymbol{\frac{x(x-2)}{x-5}}$$

Note: To score full marks, make sure you have fully factorised all expressions, and cancelled where possible. If the expressions are combined and multiplied out before being factorised, it becomes very difficult to subsequently factorise and simplify.

14 $x^{\frac{3}{2}} = 8$ where $x > 0$ and $y^{-2} = \frac{25}{4}$ where $y > 0$

Work out the value of $\frac{x}{y}$.

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$\frac{x}{y} =$ .... *(5 marks)*

Use rules of indices to rearrange each expression and find a value for x and for y, then combine.
Recall that $x^{-n} = \frac{1}{x^n}$. *Also* $x^{\frac{1}{n}} = \sqrt[n]{x}$ *and* $x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$.

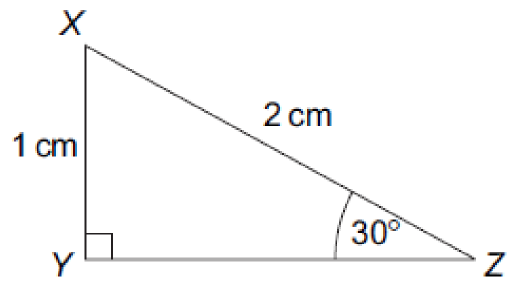
$$x^{\frac{3}{2}} = 8 \quad \Longrightarrow \quad x^3 = 8^2 = 64 \quad \Longrightarrow \quad x = \sqrt[3]{64} = 4$$

$$y^{-2} = \frac{25}{4} \quad \Longrightarrow \quad y^2 = \frac{4}{25} \quad \Longrightarrow \quad y = \sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$$

$$\frac{x}{y} = 4 \div \frac{2}{5} = 4 \times \frac{5}{2} = \frac{20}{2} = 10$$

$$\boldsymbol{Answer}\text{:} \quad \boldsymbol{\frac{x}{y} = 10}$$

15 (a) *XYZ* is a right-angled triangle.



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Use triangle *XYZ* to show that $\sin 60° = \frac{\sqrt{3}}{2}$

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(2 marks)

Use basic angle rules to identify a 60° angle, Pythagoras' theorem to find the length of the third side, and right-angled trigonometry to prove the desired result.

Recall that, for any right-angled triangle with hypotenuse c, $a^2 + b^2 = c^2$*, and* $\sin\theta = \frac{opp}{hyp}$.

$$Since\ XYZ\ is\ a\ right-angled\ triangle: \quad Y\hat{X}Z = 60°$$

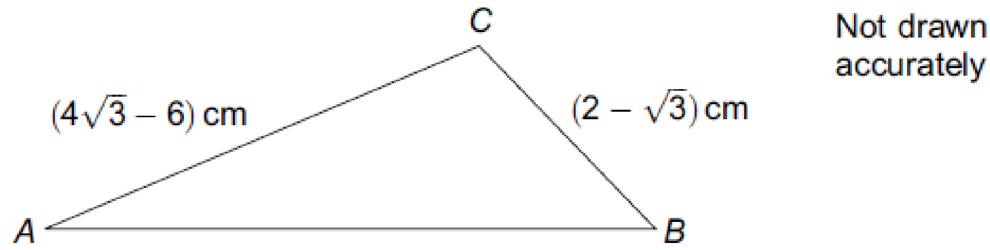
$$Using\ Pythagoras: a^2 + b^2 = c^2 \quad \Longrightarrow \quad YZ^2 + 1^2 = 2^2 \quad \Longrightarrow \quad YZ^2 = 3 \quad \Longrightarrow \quad YZ = \sqrt{3}$$

$$In\ relation\ to\ the\ 60°\ angle: \quad XZ = hypotenuse = 2 \quad and \quad YZ = opposite = \sqrt{3}$$

$$Applying\ the\ sine\ ratio\ formula: \quad \sin 60 = \frac{\sqrt{3}}{2}$$

$$\boldsymbol{Answer}: \quad (\boldsymbol{proof})$$

15 (b) Triangle ABC has an obtuse angle at C.



Given that $\sin A = \frac{1}{4}$, use triangle ABC to show that angle $B = 60°$

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(6 marks)

Use the sine rule to construct an equation, simplify (rationalising the denominator of any fraction involving surds, if necessary) and solve to find the possible angles for B. Finally use the information about the angle at C to determine the correct answer.

Recall that, for a triangle with sides a, b and c and opposite angles A, B and C respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Longrightarrow \quad \frac{2-\sqrt{3}}{\frac{1}{4}} = \frac{4\sqrt{3}-6}{\sin B} \quad \Longrightarrow \quad \sin B = \frac{\frac{1}{4}\left(4\sqrt{3}-6\right)}{2-\sqrt{3}}$$

$$\sin B = \frac{\sqrt{3}-\frac{3}{2}}{2-\sqrt{3}} = \frac{\left(\sqrt{3}-\frac{3}{2}\right)\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)} = \frac{\left(2\sqrt{3}-3+3-\frac{3}{2}\sqrt{3}\right)}{4-3} = \frac{\sqrt{3}}{2}$$

$$\Longrightarrow \quad B = \sin^{-1}\frac{\sqrt{3}}{2} = 60° \quad or \quad 120°$$

$$Since\ angle\ C\ is\ obtuse, angle\ B\ must\ be\ acute \quad \Longrightarrow \quad B = 60°$$

$$\boldsymbol{Answer}: \quad (\boldsymbol{proof})$$

Note: It is necessary to prove that $60°$ is the only correct answer by eliminating all other possibilities. If required, sketch the curve $y = \sin x$ to ensure all valid solutions are considered.

16 Prove that $\tan\theta + \dfrac{1}{\tan\theta} \equiv \dfrac{1}{\sin\theta\cos\theta}$

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(3 marks)

Start with one side, and use trigonometric identities to demonstrate algebraically that it can be written as the other side.

Recall that $\tan x = \frac{\sin x}{\cos x}$ *and* $\sin^2 x + \cos^2 x = 1$.

$$LHS = \tan\theta + \frac{1}{\tan\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = RHS$$

Note: When proving an identity you must demonstrate that one side is a rearrangement of the other. Merely manipulating the equation and not finding a contradiction is not sufficient.