Using De Moivre to turn powers of trig into multiple angles

To write $\cos^n\theta$ or $\sin^n\theta$ in terms of $\cos k\theta$ or $\sin k\theta$, use the following identities:

$$\boldsymbol{z+\frac{1}{z}=2\cos\theta} \qquad \boldsymbol{z-\frac{1}{z}=2i\sin\theta}$$

$$\boldsymbol{z^n+\frac{1}{z^n}=2\cos n\theta} \qquad \boldsymbol{z^n-\frac{1}{z^n}=2i\sin n\theta}$$

1. Write the function in terms of $z+\frac{1}{z}$ or $z-\frac{1}{z}$, then use binomial expansion to multiply out the bracket.
2. Group to produce terms of the form $z^n+\frac{1}{z^n}$ or $z^n-\frac{1}{z^n}$ and replace with the relevant trigonometrical function.

Eg:

$$\cos^3\theta=\frac{1}{8}(2\cos\theta)^3=\frac{1}{8}\left(z+\frac{1}{z}\right)^3=\frac{1}{8}\left[z^3+3\left(z^2\frac{1}{z}\right)+3\left(z\frac{1}{z^2}\right)+\frac{1}{z^3}\right]$$

$$=\frac{1}{8}\left[\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)\right]=\frac{1}{8}[2\cos 3\theta+3(2\cos\theta)]=\frac{3\cos\theta+\cos 3\theta}{4}$$

$\cos^n\theta$ *in terms of* $\cos k\theta$	$\sin^n\theta$ *in terms of* $\sin k\theta$ *and* $\cos k\theta$
$\cos\theta=\cos\theta$	$\sin\theta=\sin\theta$
$\cos^2\theta=\frac{1+\cos 2\theta}{2}$	$\sin^2\theta=\frac{1-\cos 2\theta}{2}$
$\cos^3\theta=\frac{3\cos\theta+\cos 3\theta}{4}$	$\sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4}$
$\cos^4\theta=\frac{3+4\cos 2\theta+\cos 4\theta}{8}$	$\sin^4\theta=\frac{3-4\cos 2\theta+\cos 4\theta}{8}$
$\cos^5\theta=\frac{10\cos\theta+5\cos 3\theta+\cos 5\theta}{16}$	$\sin^5\theta=\frac{10\sin\theta-5\sin 3\theta+\sin 5\theta}{16}$
$\cos^6\theta=\frac{10+15\cos 2\theta+6\cos 4\theta+\cos 6\theta}{32}$	$\sin^6\theta=\frac{10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta}{32}$

Using De Moivre to turn multiple angles trig into powers

To write $\cos n\theta$ or $\sin n\theta$ in terms of $\cos^{k}\theta$ or $\sin^{k}\theta$, use the following identity:

$$(\boldsymbol{\cos\theta + i\sin\theta})^{\boldsymbol{n}} = \boldsymbol{\cos n\theta + i\sin n\theta}$$

Write this formula with the relevant power, multiply out and simplify, then equate either real or imaginary parts, depending on whether you want an expression for $\sin^{k}\theta$ or $\cos^{k}\theta$.

Eg:

$$\cos 3\theta = Re\{\cos 3\theta + i\sin 3\theta\}$$

$$= Re\{(\cos\theta + i\sin\theta)^3\} = Re\{\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta\}$$

$$= \boldsymbol{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

Or, by substituting $\sin^2\theta = 1 - \cos^2\theta$, we can write in terms of the single function $\cos\theta$:

$$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta) = \boldsymbol{4\cos^3\theta - 3\cos\theta}$$

Note: the expression cannot always be written nicely in terms of only a single trig function.

$\boldsymbol{\cos n\theta}$ ***in terms of*** $\boldsymbol{\sin^{k}\theta}$ ***and*** $\boldsymbol{\cos^{k}\theta}$	$\boldsymbol{\sin n\theta}$ ***in terms of*** $\boldsymbol{\sin^{k}\theta}$ ***and*** $\boldsymbol{\cos^{k}\theta}$
$\cos\theta = \cos\theta$	$\sin\theta = \sin\theta$
$\cos 2\theta = 1 - 2\cos^2\theta$	$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ $= 8\sin^4\theta - 8\sin^2\theta + 1$	$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$ $= 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)$
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ $= \cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5)$	$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ $= \sin\theta\,(16\sin^4\theta - 20\sin^2\theta + 5)$
$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ $= -32\sin^6\theta + 48\sin^4\theta - 18\sin^2\theta + 1$	$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$ $= 2\sin\theta\cos\theta\,(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta)$

Note: expressions for $\tan n\theta$ in terms of $\tan^{k}\theta$ can be formed by generating the relevant $\sin n\theta$ and $\cos n\theta$ expressions, but *leaving them in the mixed* $\sin\theta$ *and* $\cos\theta$ *form*, using $\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$ and dividing through top and bottom by the greatest power of $\cos\theta$. This eliminates all instances of $\cos\theta$ and converts the $\sin\theta$ components to $\tan\theta$.