Vectors Exam Question (AQA C4 – Jan '11)

8 The coordinates of the points A and B are $(3, -2, 4)$ and $(6, 0, 3)$ respectively.

The line l_1 has equation $\mathbf{r} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

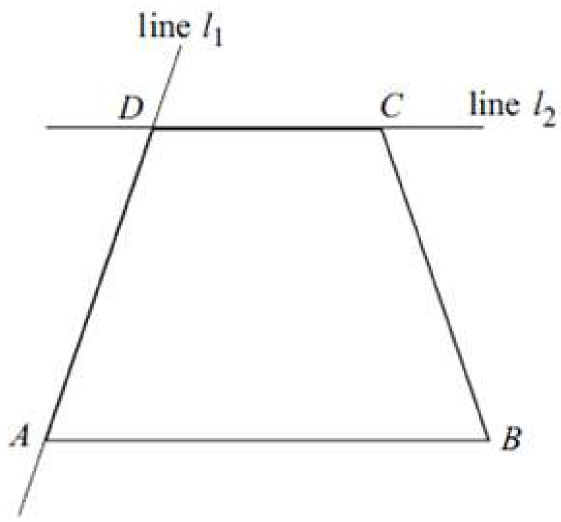
(a) (i) Find the vector $\overrightarrow{AB}$. *(2 marks)*

(ii) Calculate the acute angle between $\overrightarrow{AB}$ and the line l_1, giving your answer to the nearest 0.1°. *(4 marks)*

(b) The point D lies on l_1 where $\lambda = 2$. The line l_2 passes through D and is parallel to AB.

(i) Find a vector equation of line l_2 with parameter μ. *(2 marks)*

(ii) The diagram shows a symmetrical trapezium $ABCD$, with angle DAB equal to angle ABC.



The point C lies on line l_2. The length of AD is equal to the length of BC.

Find the coordinates of C. *(6 marks)*

Solutions

8.
a)
i.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$$

ii.
Use the dot (scalar) product to find the angle:

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} . \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \left\| \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\| \left\| \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right\| \cos\theta$$

$$1 = \sqrt{14}\sqrt{14}\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\frac{1}{14} = \mathbf{85.9}°$$

b)
i.

$$\overrightarrow{OD} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix}$$

Write a line equation for l_2 using $\overrightarrow{AB}$ as the direction vector and $\overrightarrow{OD}$ as the position vector:

$$\boldsymbol{l_2}: \quad \boldsymbol{r} = \begin{bmatrix} \mathbf{7} \\ \mathbf{-4} \\ \mathbf{10} \end{bmatrix} + \boldsymbol{\mu}\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$$

ii.

$$\overrightarrow{OC} \textit{ is on line } l_2 \quad \Rightarrow \quad \overrightarrow{OC} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} + p\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 + 3p \\ -4 + 2p \\ 10 - p \end{bmatrix} \quad \textit{for some } p$$

Use the position vector of B to find the vector $\overrightarrow{BC}$ in terms of p:

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{bmatrix} 7+3p \\ -4+2p \\ 10-p \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3p \\ -4+2p \\ 7-p \end{bmatrix}$$

Find the vector $\overrightarrow{AD}$:

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$$

$$\left|\overrightarrow{AD}\right| = \left\|\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}\right\| = \sqrt{4^2 + (-2)^2 + 6^2} = \sqrt{56}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(1+3p)^2 + (-4+2p)^2 + (7-p)^2}$$

Since the lines AD and BC are of equal length:

$$\sqrt{(1+3p)^2 + (-4+2p)^2 + (7-p)^2} = \sqrt{56}$$

$$(1+3p)^2 + (-4+2p)^2 + (7-p)^2 = 56$$

$$1 + 6p + 9p^2 + 16 - 16p + 4p^2 + 49 - 14p + p^2 = 56$$

$$10 - 24p + 14p^2 = 0$$

$$5 - 12p + 7p^2 = 0$$

$$(7p-5)(p-1) = 0$$

$$\Rightarrow \quad p = 1 \; or \; p = \frac{5}{7}$$

$$Of\ the\ two\ possible\ positions\ for\ C\ on\ the\ line\ l_2\ we\ require\ the\ one\ closest\ to\ D:$$

$$p = \frac{5}{7} \quad \Rightarrow \quad \overrightarrow{OC} = \frac{1}{7}\begin{bmatrix} 64 \\ -18 \\ 65 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{C}\text{: } \left(\boldsymbol{\frac{64}{7}}, -\boldsymbol{\frac{18}{7}}, \boldsymbol{\frac{65}{7}}\right)$$

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12/04/2013