Vectors Exam Question (AQA C4 – Jan '11)

8 The coordinates of the points A and B are $(3,\ -2,\ 4)$ and $(6,\ 0,\ 3)$ respectively.

The line l_1 has equation $\mathbf{r} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

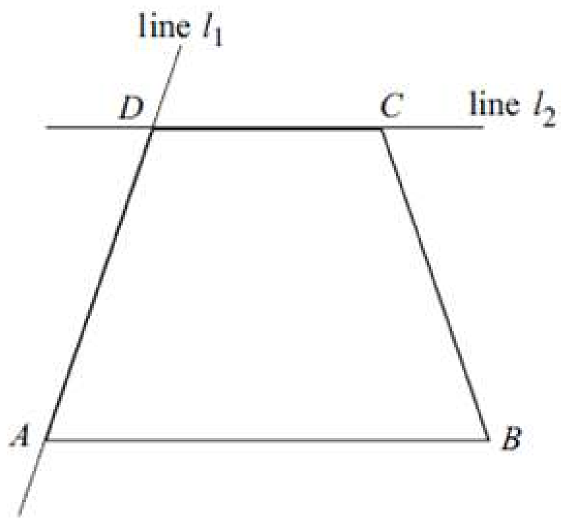
(a) (i) Find the vector $\overrightarrow{AB}$. *(2 marks)*

(ii) Calculate the acute angle between $\overrightarrow{AB}$ and the line l_1, giving your answer to the nearest $0.1°$. *(4 marks)*

(b) The point D lies on l_1 where $\lambda = 2$. The line l_2 passes through D and is parallel to AB.

(i) Find a vector equation of line l_2 with parameter μ. *(2 marks)*

(ii) The diagram shows a symmetrical trapezium $ABCD$, with angle DAB equal to angle ABC.



The point C lies on line l_2. The length of AD is equal to the length of BC.

Find the coordinates of C. *(6 marks)*

Solutions

8.
a)
i.
Apply the result from vector addition: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$$

ii.
Use the definition of the dot (scalar) product: $a \cdot b = |a||b| \cos\theta$ and rearrange to find θ:

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} . \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \left|\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right| \left|\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right| \cos\theta$$

$$1 = \sqrt{14}\sqrt{14}\cos\theta \quad \Longrightarrow \quad \theta = \cos^{-1}\frac{1}{14} = \mathbf{85.9}°$$

b)
i.
Substitute $\lambda = 2$ into a general point on l_1 to find the position vector of D:

$$\overrightarrow{OD} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix}$$

Write a line equation for l_2 using $\overrightarrow{AB}$ as the direction vector and $\overrightarrow{OD}$ as the position vector:

$$\boldsymbol{l_2}\text{:} \quad \boldsymbol{r} = \begin{bmatrix} \mathbf{7} \\ \mathbf{-4} \\ \mathbf{10} \end{bmatrix} + \boldsymbol{\mu}\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$$

ii.
Using a general point from line l_2 to find an expression for the position vector of C:

$$\overrightarrow{OC} \textit{ is on line } l_2 \quad \Longrightarrow \quad \overrightarrow{OC} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} + p\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7+3p \\ -4+2p \\ 10-p \end{bmatrix} \quad \textit{for some } p$$

Use the position vector of B to find the vector $\overrightarrow{BC}$ in terms of p:

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{bmatrix} 7+3p \\ -4+2p \\ 10-p \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3p \\ -4+2p \\ 7-p \end{bmatrix}$$

Find the vector $\overrightarrow{AD}$:

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$$

Use Pythagoras' theorem to find the magnitude of the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

$$\left|\overrightarrow{AD}\right| = \left\|\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}\right\| = \sqrt{4^2 + (-2)^2 + 6^2} = \sqrt{56}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(1+3p)^2 + (-4+2p)^2 + (7-p)^2}$$

Equate the two, since the lines AD and BC are of equal length:

$$\sqrt{(1+3p)^2 + (-4+2p)^2 + (7-p)^2} = \sqrt{56}$$

$$(1+3p)^2 + (-4+2p)^2 + (7-p)^2 = 56$$

$$1 + 6p + 9p^2 + 16 - 16p + 4p^2 + 49 - 14p + p^2 = 56$$

$$10 - 24p + 14p^2 = 0$$

$$5 - 12p + 7p^2 = 0$$

$$(7p-5)(p-1) = 0$$

$$\Rightarrow \quad p = 1 \; or \; p = \frac{5}{7}$$

Use the diagram to decide which value of p gives the correct location:

$$Of\ the\ two\ possible\ positions\ for\ C\ on\ the\ line\ l_2\ we\ require\ the\ one\ closest\ to\ D:$$

$$p = \frac{5}{7} \quad \Rightarrow \quad \overrightarrow{OC} = \frac{1}{7}\begin{bmatrix} 64 \\ -18 \\ 65 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{C}\text{: } \left(\boldsymbol{\frac{64}{7}}, -\boldsymbol{\frac{18}{7}}, \boldsymbol{\frac{65}{7}}\right)$$

Key Results Used

The **vector between points** A and B is denoted $\overrightarrow{AB}$ and can be calculated from the position vectors of A and B (usually written as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively) as follows:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

The **magnitude** of the 2D vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and the 3D vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are given by:

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2} \qquad \left\| \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\| = \sqrt{a^2 + b^2 + c^2}$$

The **scalar product** or **dot product** of two vectors is a method for multiplying two vectors together to produce a scalar. The definition is given as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$$

$$\textit{where } \theta \textit{ is the angle between the two vectors}$$

To **calculate the dot product** of two vectors is straight-forward in column form:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

The dot product is most commonly used **to find the angle between two vectors**. Since this is the case, it can be more helpful to use the following version of the definition:

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \cos \theta$$

The **vector equation of a line** is given in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ where $\boldsymbol{a}$ is the **position vector** (any point on the line) and $\boldsymbol{b}$ is the **direction vector** (any vector with the same direction as the line).

An example might be:

$$\boldsymbol{r} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$$

This goes through the point (3,4,1) and points in the direction of $\begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$

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