Teaching and Learning Mathematics at A-level:
From Algebra to Calculus

A report on A-level Mathematics within the context of the transition from Algebra to Calculus, drawing on both school experience and relevant literature.

"Mathematics is fundamentally important in an all-pervasive way, both for the workplace and for the individual citizen." This is one of the conclusions of the Smith report, Making Mathematics Count (DCSF, 2004), and it sums up the basis for mathematical education at any and all levels. There are as many reasons for studying maths as there are mathematicians, but whether we pursue it for our own enjoyment, the sake of a career, as a tool for scientific study or simply to better understand our mortgage repayments, the fact remains that maths is everywhere. The more effective our teaching of this subject is, the better equipped our students will be for whatever they go on to do.

Its very prolificacy, however, is what makes it so difficult to teach appropriately. Do we need to learn the same topics for a career in engineering as we do for a career in accountancy? Are the same skills necessary for physicists as for statistical analysts? And is there an ideal way of teaching a topic, or does that also depend on its application? Do graphic designers, for instance, need to understand the genesis of the curves and surfaces they employ? If a surveyor only ever uses right-triangle trigonometry, does he need to know and understand double-angle formulae, or be able to reproduce the graph of $y = \tan(x)$?

I would argue that it is neither the knowledge of mathematical theorems nor even adeptness in carrying out any specific mathematical operations which makes A-level Mathematics such a highly respected qualification in so many professions, but rather it is those things of which we hope our assessment may be a good indicator – a keen, inquiring mind, accustomed to meeting new and challenging problems and dealing with them in a rigorous, logical and insightful manner. It is precisely these skills which are required increasingly in every area of development from medical research to automobile design, and from computer science to engineering. In fact, as computing power increases exponentially, the tasks of recalling mathematical theorems and proofs, and even carrying out algorithms to solve standard problems can be delegated to a computer for a fraction of the cost and a substantial increase in efficiency and reliability.

Therefore, while a knowledge of mathematical theory and practice is undoubtedly of some value, it is essential that this goes hand in hand with the adaptability, mathematical intuition and problem-solving ability that makes us more than 'thinking machines'. The same principle holds true, to one degree or another, in all school subjects; students are not taught about past events for the sake of producing historians capable of regurgitating what anyone could learn on Wikipedia, but in order to develop a spirit of inquiry and a clear, impartial approach to the study of sources in order to draw valid conclusions. Some subjects are more informal in the development they engender, but even in the study of woodwork, it is not the blind ability to replicate artefacts that is the primary aim, but rather an understanding of the material's properties – 'getting a feel for it' – which will put learners in a position to create or adapt designs for manufacture. Even though the non-creative element of this process is beginning to be exploited by computers for simpler materials such as polymers and metal alloys, the design input is still very much in the domain of the human mind.

The mathematics we present in schools today is a distillation of millennia of study and investigation. It is necessarily an enormous field, and the sheer broadness of the subject is enough to daunt new learners, but we should also bear in mind that even the elements we see as fundamental and straight-forward were ground-breaking at their inception, and they will be as hard to grasp for 12-year-olds today as they were for the Ancient Greeks in their day.

Ultimately leading on to calculus, algebra has its roots in the concept of quantity, or number, with further abstraction bringing us fractions, negatives and operations such as addition and multiplication, and then exponentials and trigonometry following on. Eventually the concept of numerical operations is itself abstracted into symbolic algebra, where the values represented by a letter become increasingly unimportant in comparison with the patterns and relationships observable in the manipulation of the abstract algebraic expressions.

It takes years for us to build a competence with the manipulation of even the natural numbers, and the majority of adults will still become confused when required to work with negatives or fractions, even though these are approaching the ubiquity of positive whole numbers in modern life. Mathematics is still widely seen as the most difficult school subject, and within secondary mathematics, algebra is generally labelled the toughest topic. It should come as no surprise, therefore, that calculus, as the next big step in this hierarchy of abstraction, is a hard topic to fathom for many pupils.

Vygotsky's theory relating to the Zone of Proximal Development (Vygotsky, 1978) and the subsequent concept of scaffolding developed from it serve as valuable tools for analysing the difficulties students have with the foundational material of A-level Mathematics. GCSE students who achieve a C or above are often encouraged to consider taking their mathematical education on to A-level, but the nature of the GCSE examination means that a large proportion of those working at a grade C do not have the necessary grasp of the topics necessary for starting to tackle the A-level syllabus. The notion of scaffolding suggests that there are certain topics that must be understood before a given concept can be properly grasped, and if these building blocks are not adequately in place, or are themselves ill-comprehended, the learner will struggle to assimilate the new concept.

In the case of calculus, just as we require a working knowledge of how numbers interact before commencing algebra, there must be a concept of function in order for the basis of calculus – the study of the structure or 'algebra' of functions – to make any sense. Ideally, students should already be familiar with the graphical representations of polynomials up to order 3 or 4, as well as the more common functions such as the exponential, trigonometric ratios and the reciprocal 1/x function. The student must recognise the implications of these graphs, and be able to make links between solutions to the algebraic form and the graph, as well as a working knowledge of why a function takes certain trends. Only at this point will calculus fall within the ZPD of the student. That is, within the limit of what may be achievable through study while still being beyond their current skill level. The concept of function, and how it may be learned, will be discussed later on.

It may be argued that without some level of understanding, we are unable to interact with the world in any meaningful way whatsoever. Piagetian stages of growth begin with the Sensori-motor stage, where a child starts to respond to his or her environment, learns to manipulate it, react to stimuli in a learned fashion, and begin to grasp the concept of object permanence – the idea that objects exist independent of our experience of them (Chapman, 1988). Even at this very early stage we start to see that the child must be constructing within themselves a model – albeit a limited and imperfect one – of the world around them in order to make predictions about the results of an action. It is this working model that allows us to build on experiences and let them influence our decision-making; that is, allows us to learn, and the process never stops.

In fact, it ought to be noted that by the time a child reaches the point of studying A-level mathematics, they have been learning almost constantly for 16 years. Every new experience, every piece of information has been analysed, stored away and, depending on the value placed on it and its connection with other pieces of information of importance, may be recalled and used to inform future choices. Whether we are aware of it or not, we become very good at learning efficiently. Those things we merely need to know how to do, we can learn by repetition or simple memory techniques, and those things we find it is valuable to us to understand more fully, we learn by interacting with them, making links between related concepts and building a coherent mental image of the situation.

Skemp's categorization of understanding as either Relational or Instrumental (Skemp, 1976), while it may be seen as an over-simplification of the processes, can also shed light on how learning takes place at A-level. Many pupils find that by developing rote-learning and memorization techniques they can achieve equally good results in maths tests for less work than those who struggle to opt to make the effort to obtain a full grasp of the various elements. As Ofsted conclude in their report on provision for 14-19-year-olds:

> "... there is considerable evidence, not only from this survey, that many students are learning to pass mathematics examinations without necessarily gaining the mathematical skills and understanding to apply their knowledge accurately and independently." (Ofsted, 2006)

Certainly up to GCSE, and for some people beyond, with the narrow range of question types and topics tested in national examinations, it is perfectly possible for a heavily instrumental understanding to produce good grades. Relational understanding cannot be taught as instrumental understanding is taught. Taking the map and directions analogy, you can be directed to your destination by means of a series of instructions, and find your way quickly and efficiently. Or you can learn the layout of the area, which roads go where, important land-marks and so forth. The latter level of understanding would give you the flexibility to critically analyse your instructions, alter them for unforeseen circumstances, recover from mistakes, and create your own routes for a myriad of new destinations. However, you wouldn't expect or desire, should you stop for directions, that those directing you would attempt to instil this level of comprehension in you.

> "No wonder, there is an air of relief when they move from 'first principles' to the formal algorithm for differentiating polynomials. As one bright student said: 'It's typical of teachers to show us a lot of difficult methods before getting on to the easy way to do it'." (Tall, 1985)

This response is precisely that of an instrumental learner to a teacher who attempts to impart a relational picture in a short time. When we learn the layout of a new neighbourhood, we do it by exploring, by studying a map and by trial-and-error journeys. Likewise, when we set out to learn mathematics relationally we cannot do it by listening to a series of steps as we would with the instrumental model. We must explore the new concept, examine the big picture, see how it links in with other areas we already understand, and invest sufficient time to build the confidence that, eventually, will allow us to solve unfamiliar problems within the topic. When teaching for relational understanding, we must be providing opportunities for relational learning, not merely adding embellishment to an algorithmic and procedural method of imparting information in the vain hope that such a conceptual understanding will somehow be automatically instilled in the mind of the learner.

Skemp argues that a relational understanding – that is, a comprehension of a topic which involves knowledge of how it fits in with previous information, a bank of instances or examples of its use or representation and the ability to adapt and apply them to unfamiliar situations – is more desirable in the long run, despite being more difficult to acquire in the short-term. Certainly when tackling a new topic it is important to assess whether it is worth the extra effort to develop a relational understanding, or whether a more limited instrumental understanding would be sufficient for your purposes.

If A-level Mathematics is intended to give employers and Higher Education providers some indication of not just mathematical knowledge but the ability of students to apply this knowledge, it would seem obvious that a relational understanding is essential. The problem partially lies, therefore, with our assessment of A-level. After taking a GCSE, with its narrow proscriptive range of questions, students are not necessarily prepared for a course which requires this depth of comprehension. In fact, even should they be anxious to develop this, it is imperative that they start by thoroughly understanding the content of the GCSE syllabus. Even at A-level, too often we demonstrate a method to be used for a particular question, then a new method for a slightly different question, and ultimately may be giving the message that simply by memorizing these algorithms students can become A-level standard mathematicians. To engender effective learning it is imperative that we provide circumstances and situations in which pupils can explore new topics, construct conjectures and critically examine each other's, and their own, theories. It is only when this deep relational understanding is beginning to develop that students can begin to truly use and apply their knowledge, not just to examination questions of a set form, but to applicable problems they may encounter in the future.

> "The predominant didactic teaching style which is oriented towards developing and practising techniques, and then applying them to structured standard problems, may be successful in enabling students to obtain the examination grades they require. However, it seems to be much less effective in developing both the understanding and fluency and the thought and persistence that are needed for successful problem-solving." (Haggarty, 2002)

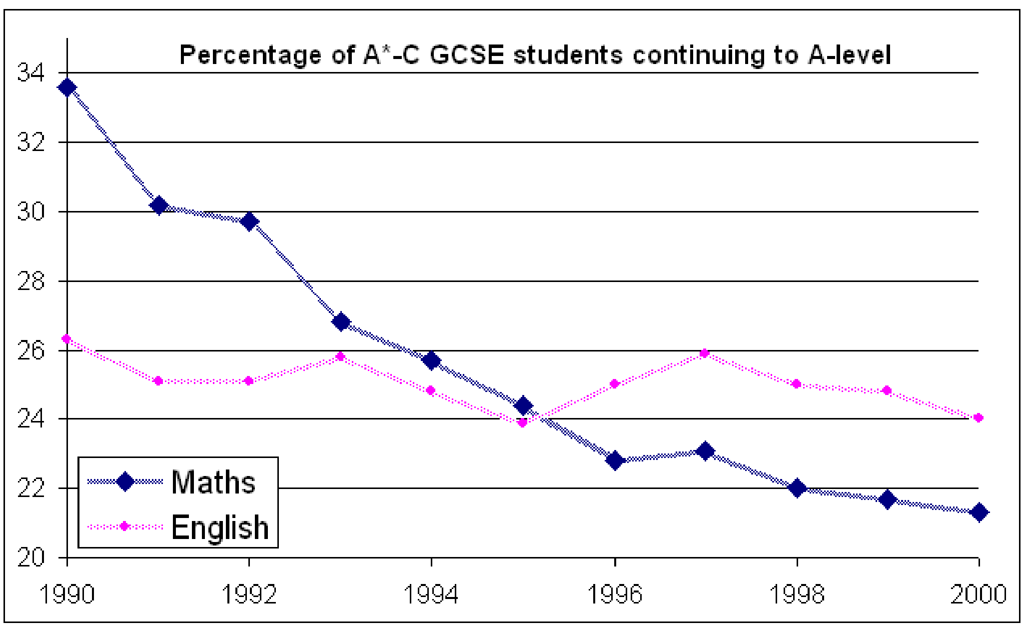
It could be argued that ultimately the learner is responsible for developing this type of understanding for themselves. The theory of Constructivism is based around the claim that "Knowledge is not passively received, but is actively built up by the cognising subject" (Glasersfeld, 1983). However, the Ofsted report on 14-19 Mathematics provision found that "The quality of teaching was the single most important factor influencing students' achievement." This report identifies a number of factors which, where in evidence, were closely linked with improved performance in learners in terms of depth of understanding. Subject knowledge was the first factor identified, and although it may seem obvious, in the current economic climate schools may well find it difficult to find mathematics teachers sufficiently qualified to confidently deliver the A-level syllabus. This is even more of a problem for Further Maths, although in an increasing number of schools the Further Maths Network is providing expert coaching for groups of pupils from up to half a dozen schools together.

The report promotes a focus on critical thinking and reasoning, specifically encouraging "a spirit of collaborative enquiry that promotes mathematical discussion and debate." This ties in with their next point, which is that of questioning and pace. Teachers often struggle with the sheer weight of content at A-level, but it is essential to get the right balance between moving ahead in the syllabus and allowing time – structured and directed through carefully considered, thought-provoking questions – for learners to explore new concepts and start to develop their own understanding. It may take slightly longer to work through topics, but this is more than compensated for by learners being equipped with the tools they will need not only later on in the course or in further study, but as they go on to use maths in their careers, to apply knowledge and understanding appropriately to new and unfamiliar situations. The report's recommendation to the DfES and QCA includes a revision of GCSE and A-levels in "examinations which encourage effective understanding and problem-solving, as well as competence in mathematical techniques." (Ofsted, 2006)

Clearly, this depth of understanding is an extremely difficult thing to quantitatively assess, and it may reasonably be assumed that any nationally standardised test must perforce be a compromise between questions of sufficient variability to test a student's understanding beyond application to familiar situations and questions of a style and type capable of producing comparable results across the nation through a reasonable level of external moderation. The question of assessment is an issue of huge importance, but it does not fall under the scope of this report except insofar as it pertains directly to the way A-level Mathematics is taught and learnt.

How does the way we assess A-level affect the way we teach it? Assessment can only ever be an imperfect model of how the mathematical understanding of the student may be used in the future, but the closer it comes as a predictor of success in either further education or a career, the better we can assume it to be. It is for this reason that A-level papers are notoriously harder to mark than, for instance, GCSE. Instead of looking for a basic grasp of fundamental numerical facts and ideas we are examining for a much broader and deeper level of all-round mathematical prowess. Unfortunately, the scope of the A-level syllabus rarely leaves enough time for teachers and students to focus on the development of the underlying skills, and rather presses ahead with new topic after new topic, encouraging, if anything, the idea that learning by rote is the best way to cope with the material.

A-level mathematics requires considerably more than GCSE, and although a large number of schools still allow entrants with a C grade or above, the demands of the course mean that many schools now only take A*, A and B grade candidates. This may contribute to a drop over the last two decades in the percentage of A*-C GCSE students who continue to A-level, but it is unlikely to be the only factor. The following graph plots data from the QCA on the relative percentages of English and Maths GCSE students taking on A-level:



(Haggarty, 2002)

Mathematics at the post-16 level is by no means limited to AS and A2 level qualifications, and it is evident from the increasing number of alternatives becoming available that the educational research community are less than satisfied with its relevance and appropriateness. Despite mathematics being an integral part of a wide range of specifically vocational courses, there has been some concern at the absence of a GNVQ in Mathematics. (Haggarty, 2002) In response, the QCA has developed FSMQs (Free-Standing Mathematics Qualifications), which cover the first three NVQ levels. It is hoped that students will find these a user-friendly alternative to a GCSE retake, and will have more of an applied and applicable flavour than much of the current GCSE syllabus.

Those students currently starting secondary school in year 7 will be required to stay on in some form of education or training beyond the age of 16.

> "Functional Skills qualifications in English, mathematics and ICT will be taken alongside reformed GCSEs, the new Diplomas, Foundation Learning Tier and Apprenticeships, as well as being available as standalone qualifications for young people and adults." (DCSF, 2008)

When considering the provision of mathematics for post-16, the recent legislation regarding the school leaving age and the rapidly widening range of qualifications available to mainstream schools must be taken into account. A-level mathematics is being rapidly sidelined in many schools and colleges, if not abandoned altogether, in favour of the International Baccalaureate. This globally recognised qualification is offered as part of a programme of study which compulsorily includes subjects from the main subject groups; English (or First Language), Second Language, Individuals and Societies, Experimental Sciences, Mathematics and Art. (International Baccalaureate Organisation, 2009) Students complete six subjects, of which at least three must be studied at Higher level, the rest at Standard level. In addition to this, they are required to complete an Extended Essay, a Theory of Knowledge exam and a Creativity, Action, Service programme. These additional components serve to further broaden the scope of the education programme.

I ran a series of small group help sessions for year 12 AS Maths students where the intention was to build confidence with basic differentiation of polynomials, moving on subsequently to simple trigonometric functions. The lesson was an additional support option, so only students who struggled with this topic were present. It rapidly became apparent that students did not feel confident with the concept of function, and as a result had no basis in which to root their study of calculus. The instrumental rote method of learning was working well enough for those who were happy to memorise 'bring the power down in front and decrease the power by one', but even this has severe limitations which hinder progress beyond polynomials. Some students articulated a desire to learn the method without understanding it, and the rest, rebelling against this idea, tried and – for the most part – failed to incorporate it into their existing schema. As a result, the class were variously bored with going over the material or disillusioned and unmotivated with regard to the entire topic.

It was necessary, therefore, to take the whole class back to the fundamentals of algebra and the concept of function, finally demonstrating differentiation then in terms of a slope function for the graph of the original function.

APOS theory, standing for Action, Process, Object, Schema, gives us a framework which may serve to illustrate the stages involved in developing from numeracy through algebraic competence towards an understanding of function (Dubinsky & Cottrill, 1992):

Action: We see finding the square of a number as an action, or something we do to a number. For instance, we know that 3^2 is 9, and whenever we are given a number to square, we simply multiply it by itself.

Process: At this point we can write down 3^2 without 'working it out'. It is seen as something we recognise as a process we can carry out. We can see how it would be worked out, and some properties of the function, such as the square always being positive.

Object: As an object, we can now look at x^2 in its own right instead of thinking of what we get when we multiply the number that is x by itself. x^2 is something that can be manipulated – it is a number just like x was, though we know that it is always positive. We will have a clearer idea of what it means to square a number, such as $x^2<x$ only for $x<1$, and recognise that as x increases, x^2 increases faster. We could also have some concept of quadratic equations.

Schema: Finally we have a comprehensive understanding of x^2 in terms of what it does and how we can manipulate it. As part of the image attached to it, we may have a parabolic graph. We might include ideas of how quadratics behave, and how to perform transformations on the graph. We may even see how the solutions of quadratics link to a geometric square.

This schema must be in place before any of calculus can make sense to pupils. In the support class we examined how the gradient of different graphs varied along their domain, starting with constant and linear functions, demonstrating the reason for polynomial functions' gradients to be of lower order than their anti-derivative. What were once meaningless rules finally took on meaning – they were the algebraic representation of trends already observed and noted in graphical form. The derivative function for sine was created by moving along the curve estimating the gradient numerically at each point, plotting the resultant curve and comparing it to cosine.

Even a brief glance at the syllabus gives us some idea of the range of different 'rules' and techniques students are expected to master. In addition to the standard results, including a range of trigonometric functions, the exponential and the logarithm functions, A-level students must be able to use and apply the chain rule, the product rule, the quotient rule, know how to differentiate an inverse function, and be able to differentiate functions defined implicitly (Berry, Hanrahan, Porkess, & Secker, 2004). For a student who has struggled through GCSE, managed to attain a low C-grade by memorizing step-by-step methods, even possibly for higher level work such as solving simultaneous equations and factorising quadratics, having properly grasped very little of it, being confronted with this topic, and its daunting array of rules and techniques, is understandably overwhelming.

In their report on Evaluating Mathematics Provision for 14-19 year olds, Ofsted reports:

> "The majority of the teaching seen was at least satisfactory in preparing students for examinations. However, in promoting a really secure understanding of mathematical ideas, in stimulating students to think for themselves and to apply their knowledge and skills in unfamiliar situations, the picture was less encouraging." (Ofsted, 2006)

Three main areas of pupil development are identified here as requiring additional emphasis in teaching; a secure understanding of ideas, a focus on independent thinking, and building a confidence in the application of knowledge to new situations. Clearly these areas are much harder to assess through the national examinations, and it is perhaps for this reason that they do not have the emphasis they deserve in the A-level syllabus. The other thing they all have in common is that they are all primary characteristics of the relational understanding described by Skemp (1976) .

To assume that students do not desire a relational understanding is very unhelpful. Unfortunately, much of the experience they may have received throughout their school experience will have given the impression that mastery in maths depends upon the memorisation of formulae and being able to recall and use procedures and rules. Glib responses like “two negatives make a positive” and “the rule is: flip it and multiply” are tempting for teachers to use, especially in a room of 30 or more pupils all struggling to take on board a new concept, but in the long run they reinforce the idea that maths is about being good at blindly recalling and using facts and results that are understood poorly if at all. As teachers we must be constantly reminded, and constantly remind learners, of the reason behind pursuing understanding, both in a general sense and specifically for each new topic. Newton explains that what makes understanding a worthwhile goal is, “its flexibility in application, its durability, the way it facilitates further learning, and the way it enables critical abilities.” (Newton, 2000)

I gave some personal tuition on introductory calculus to a post-graduate whose mathematical education had stopped a number of years previously. Initially, as explained earlier, the concept of function was revisited in order to cement a base upon which to build the new material. The approach I took was to use the mechanics of motion to introduce displacement-time graphs, and then to examine what we mean by speed or velocity, and study the relationship between the descriptive, algebraic and graphical interpretations of the functions described. For instance, the motion of a vehicle that starts from rest, accelerates at a constant rate to a fixed speed, and finally decelerates to rest once more. From the information, displacement-time, velocity-time and acceleration-time graphs can be produced. The student found it a valuable exercise to come up with different scenarios which would fit the velocity-time and acceleration-time graphs, noticing how information is lost when we change from one to the next, and yet how each subsequent graph may be constructed from the information given by its predecessor.

This method of introducing the concept of calculus is helpful for a number of reasons. Firstly, it provides an illustration from real life to relate all of the examples to. This means that, where a student may ordinarily be unclear about what a first or second derivative tells us, by restricting our observation to commonly observed events we give meaning to the resultant graphs, and the student can interpret each graph independently in order to convince themselves of the validity of the methods used to produce them. Secondly, we have provided a motivation for investigating the topic which can be very difficult to find if pupils begin by practising differentiation on arbitrary 5^{th} order polynomials. Thirdly, students have had the opportunity to put together a coherent picture of what it means to differentiate and to integrate, with a general idea of how – and, crucially, some understanding of why – information is lost when we differentiate.

It is important to note that this overview of calculus is by no means a rigorous definition, or even a representative view of the scope of the field. What it does provide us with, however, is one of no doubt numerous entry-points into the study of calculus which students can use as an aid to memory, a hook for future information and theorems, perhaps a familiar model upon which to test new-found conjectures and a base which, we hope, will serve to inspire interest, and soon be embellished, extended and modified as the journey through calculus progresses. Many of the facts which are commonly presented as properties of the process of differentiation or integration, as well as common results, find in this model not only a worked example, but also an explanation by way of the link to a physical situation. It isn't difficult to calculate the gradient of straight lines, and it would be comfortably within the proximal development range of students to come up with numerous examples themselves of displacement, velocity and acceleration for a given situation, and if necessary graphing software could be utilised to make finding the algebraic forms simpler. It is crucial, a the early stages of this new topic, not to put students off with functions which they already struggle to comprehend, so a well-understood example they construct themselves can serve as a much-needed foot-in-the-door for the study of calculus. Then, and only then, is it necessary to formalise the basic results such as $f'(x) = 0$ when $f(x) = a$, and even these should be discoverable by exploratory investigation, making them more memorable for the students in the process.

> "If students are to become good problem solvers they have to be given actual problems to solve themselves, without being told what to do at each step, and they require an understanding and fluency with a range of appropriate algebraic and other skills." (Haggarty, 2002)

While A-level may not be the best framework within which to develop the skills that are, to a greater and greater extent, becoming requirements in the workplace, it remains the most popular post-16 mathematics qualification for those intending to continue into higher education, and can, if taught well, provide an invaluable basis upon which to build as well as the training and experience necessary to do so.

We cannot avoid the fact that, as the Smith report says, "mathematics has a claim to an inherently different status from most other disciplines." (DCSF, 2004) It is argued that maths, as a fundamentally abstract language, forms the basis of a huge range of crucial industries and areas of study. Increasingly, it appears that A-level is not providing adequate preparation for this. As a result, it is encouraging to see the wave of new initiatives and research surrounding post-16 mathematics. The topics chosen and the methods employed at this stage, we must bear in mind, are only part of the solution. By the time students begin A-levels or the IB most will have been studying maths as part of the school timetable for eleven years, and if we are to promote the kind of proactive inquisitive understanding-based learning that has been discussed, the years leading up to post-16 education are as crucial as those that follow, and represent an enormous opportunity. Initiatives such as the Functional Skills qualifications are intended to bring the Using and Applying Maths aspect of the GCSE to the forefront, and although there is considerable controversy over the way delivery and assessment of this will be managed, it is encouraging that such value is being placed on the application and understanding of mathematics as a tool for problem solving as opposed to merely a series of procedures and rules to be memorised and regurgitated for an exam.

> "Good learning will only occur if it is orchestrated by good interactive teaching… mathematics-in-practice is not mathematics, it will have been transformed by the practice into a form that is unrecognisable as mathematics to students. This implies that teaching has to emphasise this transformation and the relationship between mathematics and mathematics-in-practice." (Haggarty, 2002)

Ultimately the teacher is responsible for providing an environment in which the student can learn to learn. We are not teaching maths for its own sake for the majority of students, and it would be impossible within such a broadly applicable subject to teach the techniques and methods through which maths is used in every area of industry and commerce. Like the linguist who must understand the fundamental structures of spoken communication in order to apply their knowledge to the study of any given language, we must be producing learners who can apply their knowledge, and employ their models of understanding, with the help of a broad experience of diverse problem solving, to make mathematics work for them in whatever field they find themselves.

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