This document is intended to examine the background of Constructivism, specifically with respect to education, and to see what insight it gives us into the subject of learning mathematics. I believe the debate over the usefulness of Constructivism is fundamentally linked with the concept of relational understanding, and as such I will examine the various merits and drawbacks of learning mathematics in this way. I intend to describe the practicalities of learning in a constructivist framework and draw conclusions on the helpfulness of Constructivism in the learning of mathematics.

First it is necessary to define what we mean by Constructivism. This is a philosophical concept which addresses how we amass knowledge or understanding. It is made up of two statements, the first of which is the only one embraced by what is known as Trivial Constructivism, while both together define Radical Constructivism. Ernst Von Glasersfeld puts them in the following way:

1. Knowledge is not passively received but is actively built up by the cognising subject.
2. The function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality.

(1983)

I would suggest that in the context of learning mathematics, the distinction between the two types is unimportant, since it is primarily the first statement which affects the way we think about learning and teaching. Whether or not Mathematics is an ontological reality in its own right outside our own minds is not particularly relevant to how we learn the subject.

This standpoint lends a very interesting perspective to how we think about the learning of mathematics. Firstly, it is clear that, since building up knowledge is an active process, the primary role of a teacher should be to provide experiences which enable the learner to adapt and expand their own construct of reality, rather than merely presenting new information. I will try to examine the consequences of Constructivism more thoroughly in the following paragraphs.

It is important to recognise how Constructivism was devised and where it has its roots. It seems to have grown from a Piagetian view of learning. Piaget came up with four stages of growth which tally well with the idea of Constructivism. He labels them as follows:

- Sensorimotor stage
- Preoperational stage
- Concrete operational stage
- Formal operational stage

The first is the transition from sensory information being merely a disconnected flow and being rationalised by the construction of ideas such as permanence (things still exist when out of sight) and perspective (things retain their characteristics regardless of where we view them from). This is something we do normally without even considering it, but it is, in a fundamental way, the basis of our constructs.

The second stage develops this further by bringing in representation of what is perceived. Learning to talk is one example – we use a series of sounds to represent objects we see or actions upon those objects. Using this representation, children can respond to speech by quite complex procedures. For example, "Pick up the blue ball and bring it to me" involves the identification of a round object, distinguishing it from similar objects by a specific characteristic (colour), and moving it from one place to another. Along with representation comes a better understanding of shape and size as objects are manipulated.

The period of concrete operations refers to the point at which the child will begin to carry out logical operations on objects. Instead of randomly trying to fit objects through a hole, they will examine the properties of the objects, choosing likely candidates based on shape and size.

The period of formal operations is possibly the most relevant to Constructivism, since it concerns the way a child will use their brain to 'imagine' procedures and objects in order to solve a problem without reference to the objects themselves. The child can begin to consider "what if..?" questions, and use the symbolism they have learnt to come to reasoned outcomes. Even adults will find talking out loud is an aid to understanding, since it is one of the most powerful tools of symbolism we have. It is vital to note that this fourth stage cannot come about without the basis of the previous three. A child cannot begin to consider hypothetical situations with objects and procedures with which they are not familiar, and similarly cannot work logically with them until they have created a form of representation.

According to Piaget, a child is becoming confident with formal operations around the age of 7, which, investigations have shown, is the age at which their brains are developing faster, and they are learning faster than at any other time in their lives. I believe this has to do with the way they are reacting to new experiences and creating their own constructs of the world. Before this age, they are not properly familiar with formal operations, and beyond their ideas and constructs will be more difficult to change. While they are constantly being fed new experiences which they can analyse and draw conclusions from, their construct is constantly being changed and adapted.

Whenever we have new experiences, we tend to do one of three things; either we distort what we seem to have experienced to fit in with our existing ideas, or we ignore the experience altogether, or we can alter our construct to accommodate the new experience. If I see an arrow shot at close range, I might assume it travels in a straight line. My new experience might be to attempt a long range shot, and notice that the arrow lands lower than expected. I could distort the experience by assuming I must have aimed too low; I could ignore the experience, possibly by suggesting the arrow wasn't straight or there was something wrong with the bow; or I could examine the experience truthfully and maybe expand my range of experiences by trying different ranges and different angles of elevation. By assimilating these experiences, I could accommodate them in my construct by altering my image of the trajectory of the arrow from a straight line to a parabola, and do considerably better.

Of course, if we need to use the results of some experience which we haven't assimilated with our own construct, we often create a different construct. This need not be conflicting with the rest of your mental image, but it isn't necessarily compatible either. For example, when we learn how to solve n simultaneous equations in m variables, we might use a matrix and produce a result from that. However, if there were two equations in x and y, we would go back to a method we used at GCSE or A-level. The two methods are both valid, but may not be linked in the construct of the individual.

It is important to not the links between the idea of Constructivism and various other ideas introduced by mathematical educationalists. For instance, the idea of a Concept Image and Concept Definition (Vinner, 1983: 293) distinguishes between what is known as the formal definition of a new concept, and the image or construct associated with it in the subject's mind. This idea may seem a little simplistic – we can, for instance, create temporary concept images, or build up more than one concept image for the same object presented under different conditions, but it links well with Constructivism since it concerns the building up of an image to represent all you know about a particular topic.

Another useful way of looking at our construct is in terms of three types of mental structure: Frame, Script and Network. The frame is similar to the concept image; it is the set of expectations about an object or procedure – everything we associate with the concept in our heads. The script is similar to the concept definition; not necessarily the same as the formal definition, but the boundaries a concept is confined to as perceived by the individual. The network is then the set of links between different concepts, and how ideas relate to each other. Just as our frame is not static, but changes with new information and examples, so the network alters as we assimilate new ideas. When we first learn algebra, and solve simple simultaneous equations, there is no link whatsoever to the x and y we see on a graph, yet later on we can see how a plot of each equation would result in lines in the x-y plane, and how that links to a set of equations having no solutions, a finite set of solutions, or an infinite number of solutions. The concepts of a function and simultaneous equations have been linked by the network.

Another important concept for understanding how Constructivism links to the learning of mathematics is the process-object theory, referred to as APOS theory (Dubinsky, 1991). This seems to take the idea of concept definition and concept image even further. Essentially, it suggests that an Action we perform becomes a Process we can repeat, and later an Object we can examine and involve in further computations, until finally a Schema, which is similar to the concept image discussed earlier – a comprehensive set of ideas about the concept which, taken together, make up all of what we understand by it, including examples and images that help us to visualise what it means. To understand this properly, I will use the example of x^2:

Action: We see finding the square of a number as an action, or something we do to a number. For instance, we know that 3^2 is 9, and whenever we are given a number to square, we simply multiply it by itself.

Process: At this point we can write down 3^2 without 'working it out'. It is seen as something we recognise as a process we can carry out. We can see how it would be worked out, and some properties of the function, such as the square always being positive.

Object: As an object, we can now look at x^2 in its own right instead of thinking of what we get when we multiply the number that is x by itself. x^2 is something that can be manipulated – it is a number just like x was, though we know that it is always positive. We will have a clearer idea of what it means to square a number, such as $x^2<x$ only for $x<1$, and recognise that as x increases, x^2 increases faster. We could also have some concept of quadratic equations.

Schema: Finally we have a comprehensive understanding of x^2 in terms of what it does and how we can manipulate it. As part of the image attached to it, we may have a parabolic graph. We might include ideas of how quadratics behave, and how to perform transformations on the graph. We may even see how the solutions of quadratics link to a geometric square.

It is important to note that transforming the concept of x^2 into an object or schema is not necessary in order to use it. Quadratic equations can be solved by memorizing a formula or a method of factorization. This is known as instrumental understanding (Skemp, 1972), and although it is a valid way of learning, it does not progress through the stages described above, and hence does not give a full and comprehensive relational understanding:

> When a mathematician says he understands mathematical theory he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it, and what is worth remembering. He has an intuitive feeling for the subject, how it hangs together, and how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them.
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> (Michener, 1978)

It is this kind of understanding that creates the best kind of mental construct. If we can learn to understand a concept to the extent Michener describes, we can then move on, using our knowledge as a base for further mathematical thinking. We will recognise how a new idea links with previous ones, and be able to explore potential links with concepts we already understand.

If a topic is learnt relationally, branch-off topics are much easier to learn, since they can be added to an extended schema. With a primarily instrumental understanding, each new topic is learnt in isolation, with its own specific method. For example, an instrumental understanding of how to calculate the height of a tree from distance and angle of elevation will work fine for that example, and be quicker to learn, but a relational understanding of trigonometry of a right triangle will enable the learner to expand their knowledge of this problem to, say, calculating the length of rope needed to brace the tree to a given point on the ground. With a merely instrumental understanding, every new version of the problem will require a different method which has to be learnt. Method learning is open to mistakes, and without a relational understanding to give the whys and hows, they will not be noticed. If you understand how a method has been generated, you can correct inaccuracies in your own memory, and you can even adapt a method to new variants of the same problem.

We must be aware that however desirable a relational understanding might be to the pursuit of mathematics, there are disadvantages to learning in this way compared to an instrumental stance. Relational understanding is much harder to come by. Learning a formula or method without understanding its roots will be simpler, and it will get you the right answers much faster. What can be said for a relational understanding is that it is easier to remember. It is possible to learn all the formulas you need to calculate the area of common shapes for an exam, and do just as well as a relational learner, but a year or two on the instrumental learner will have forgotten some, and time will have bred inaccuracies in others. The instrumental learner may or may not take the time to memorize the formulae, but they will understand how each one is a direct result of the area of a rectangle, and hence will be able to construct them when needed at any time. Although it has taken longer to learn, the knowledge will be much more permanent. Another point in favour of relational understanding is its adaptability to new tasks. Taking the example of basic shape areas, an instrumental learner may need to find a book to tell them a new formula for finding the area of a hexagon, while the relational learner can use a similar method to that used to deduce the area of a pentagon, and come up with the formula independently. Also, relational knowledge can be thought of as a goal in itself. The feeling all mathematicians are familiar with when a new topic finally ‘clicks’, and the enlightening which enables you to see how all the theorems and examples go together is very satisfying, and, as I explained, is crucial to enable branching out into new fields.

This last point is one of the more important points in the argument for Constructivism as a way of thinking. Constructivism enables us to see how the way we assimilate new information affects the way we learn new concepts. Particularly in maths, by creating a comprehensive concept image and cultivating a relational understanding, we can ensure a full knowledge of the new concept, a good idea of the network between it and others, and generate a firm basis on which to build for further, more advanced, mathematical thinking. The process of learning mathematics is all about assimilating information, reifying it into a schema, and then building up new knowledge and understanding from these newly constructed building blocks. An appreciation of Constructivism and the active role of the learner in this task is a very useful tool in the learning of mathematics.

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