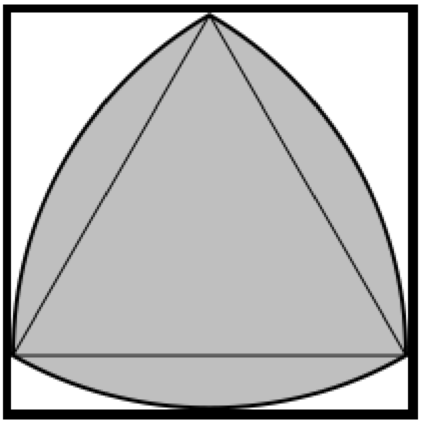
Shapes of constant width

If you are asked to name a shape that has a constant distance between the top and the bottom no matter how it is rotated, chances are you'll think of a circle. But it's not the only one…

A Reuleaux triangle is an example of a shape of constant width – that is, when sandwiched between two parallel lines, the distance between the lines is fixed no matter how the shape is rotated.

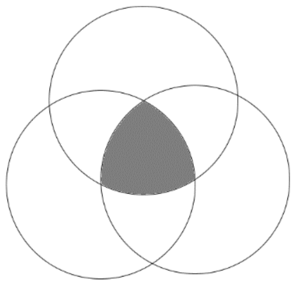
A consequence of this is that the shape also fits inside a square in such a way that it is touching each edge of the square at the same time, and can rotate within this square.



To construct this shape:

1. Draw a circle. *Keep your compass setting the same for the next circle.*
2. Draw a second circle with centre on the first circle *and the same radius*.
3. Draw a third circle *with the same radius*, whose centre is on one of the crossing points of the first two circles.

The overlap of all three circles forms a Reuleaux triangle.



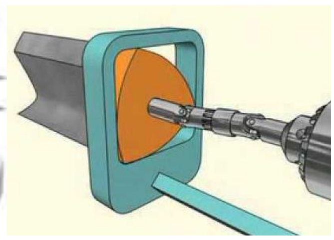
Note: joining the three corners would form an equilateral triangle.

A rotary combustion engine uses this shape as a more efficient method than pistons.

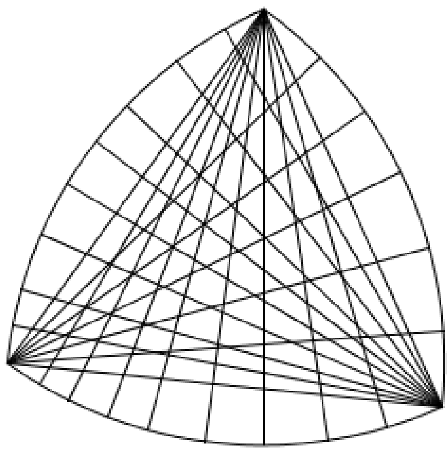
http://youtu.be/6BCgl2uumlI

A drill with a Reuleaux triangular cutting bit can drill a square(ish) hole.

http://youtu.be/L5AzbDJ7KYI



The constant width of this shape can be verified by measuring from each corner to the opposite (curved) side. Each of these lines should be the same length.



Extension: Use your knowledge of equilateral triangles and circle formulae to calculate the perimeter of this shape.
Compare this to the width of the shape (your radius). What do you notice?

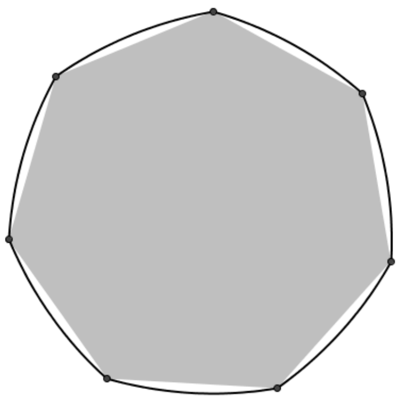
More shapes of constant width

The Reuleaux triangle is only one example of a shape of constant width.

Any polygon with an odd number of sides can be modified to generate a shape of constant width.

The heptagon opposite has had curves added, centred on the opposite corner, to form a seven-sided shape of constant width.

Do you know any everyday objects with this shape?

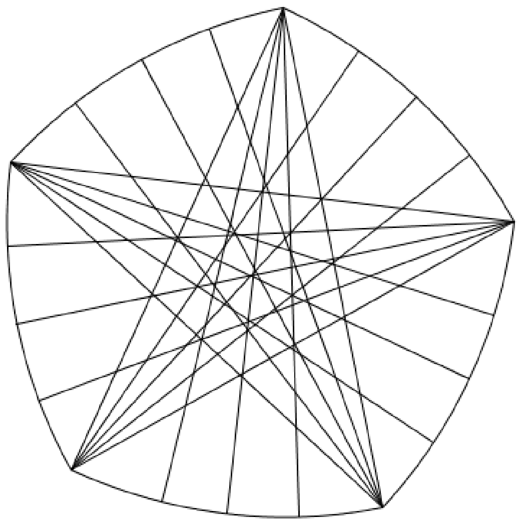


To construct a regular shape of constant width:

1. Draw a regular polygon with an odd number of sides. *This is most easily done by drawing a circle, then making marks around the circumference equally spaced (by dividing* $360°$ *by the number of sides) for the corners.*
2. Use a compass to construct an arc from one corner to another, *centred on the corner opposite*.
3. Repeat until you have used each corner as the centre for an arc, and there is an arc from each corner to the next.

The constant width of your shapes can be verified by measuring from each corner to the opposite (curved) side.

Each of these lines should be the same length.



Use your knowledge of angles in regular polygons to calculate the perimeter of your shapes.
Compare this to the width of the shape (your radius). What do you notice?

Extension: Why does this method only work for polygons with an odd number of sides? What happens if you try to turn a square or hexagon into a shape of constant width?

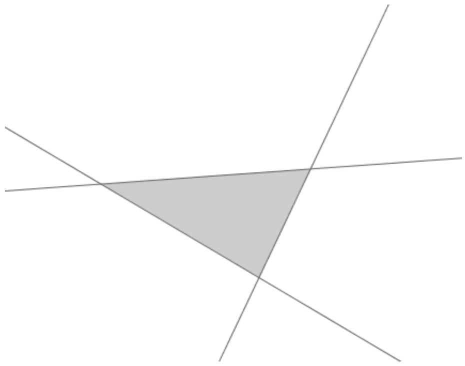
Irregular shapes of constant width

So, lots of shapes have constant width, but they're all nice and regular, right?
A constant width doesn't automatically mean a circle, or even a regular shape…

To construct an irregular shape of constant width:

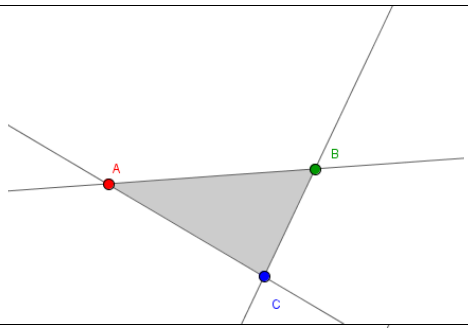
1. Draw a triangle.
   This can be equilateral (for a modified version of the Reuleaux triangle) or isosceles or scalene.

   Extend the lines of the triangle beyond the corners out across the page.



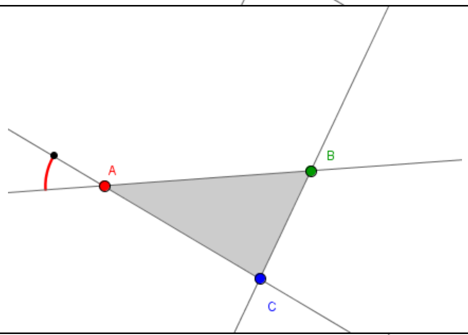
2. Label the corner opposite the shortest side A.

   Label the other two B and C, working your way clockwise from A.



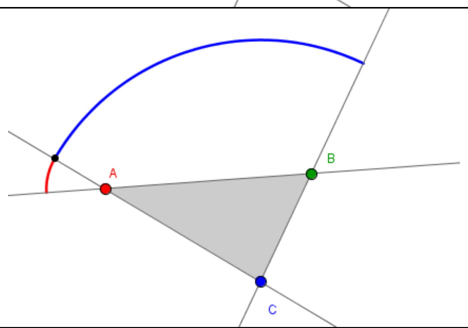
3. Choose a radius for your first arc – this can be anything from 0 up.

   Make an arc centred at A between the extended lines BA and CA.



4. Make a second arc, centred at C, with radius set so that it meets the first arc at the line CA.

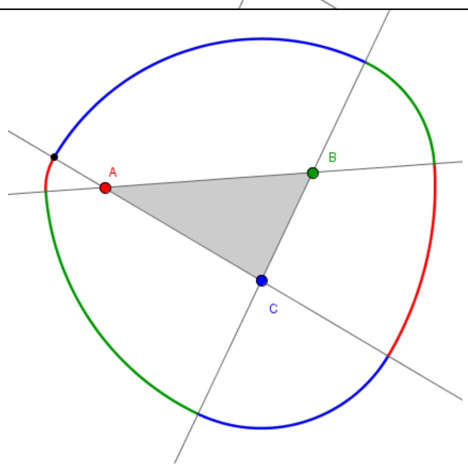
   Extend this arc to the line CB.



5. Repeat this process for the next 4 arcs.

   Make sure the centre for each arc is the point where the two lines cross.

   Extension: Measure the lengths and angles of your triangle (and the radius of the first arc), and use the cosine rule to calculate the perimeter of your shape. Compare it to the diameter. What do you notice?



Shapes of constant width SOLUTIONS

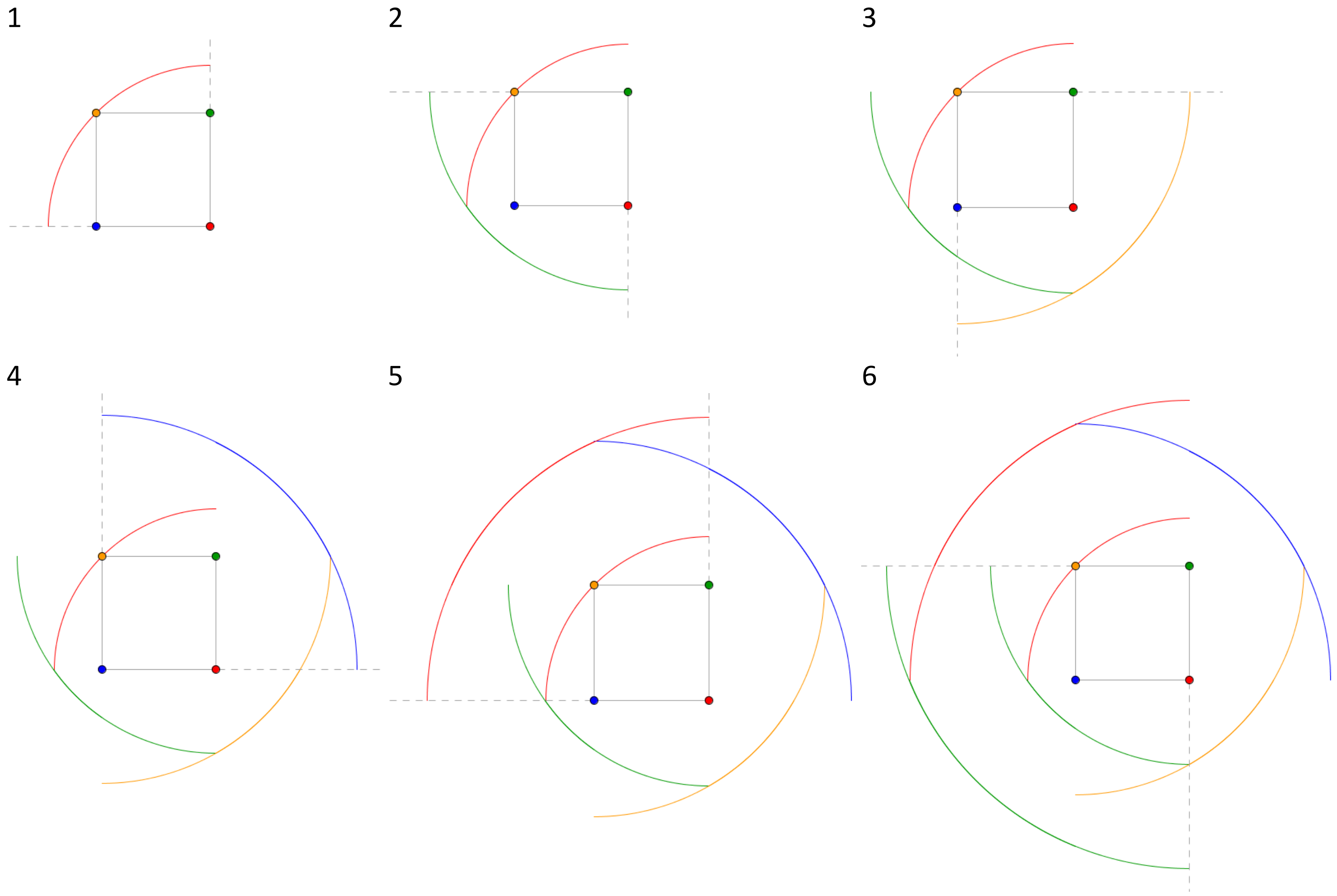
Perimeter:
To find the perimeter of a regular shape of constant width such as the Reuleaux triangle, it is necessary to calculate the angle of the sector for each arc.

The triangle itself is rather more straightforward than the pentagon or other polygons, since the angle is the same as the interior angle of the shape, $60°$.

For a Reuleaux triangle of side length x, each arc has length $\frac{60}{360}(2\pi x) = \frac{1}{3}\pi x$ and the total perimeter is therefore πx. Note that the 'diameter' of this shape (the distance between parallel lines containing the shape) is constant, and, in this case, is equal to x. Therefore **the perimeter of the Reuleaux triangle is identical to the circumference of a circle of the same diameter.**

This result is also true for shapes of constant width with a greater number of sides, and even modified versions or irregular versions, although use of non-right-angled trigonometry is required for more sides.

Even number of sides:
When constructing a shape of constant width, the procedure involves identifying a point the shape may balance on and constructing the curve required when the shape rests on this point. Then at the ends of this constructed curve, subsequent curves can be generated from other corners (see instructions for constructing an irregular shape of constant width). However, if you attempt this method with a polygon with an even number of sides…



… the minimum curve generated from the first corner provides the minimum size for the next curve, and so on, but by the time you get back round to the first curve, instead of meeting up they just spiral outwards.