Rolling Projectiles

A marble is balanced on top of a bowling ball.

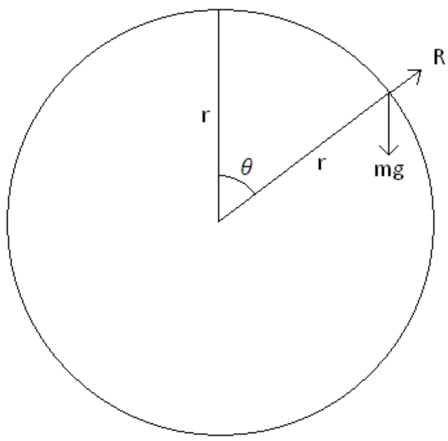
Initially at rest, it begins to roll down the ball until it leaves the surface and continues its motion under the influence of gravity alone until it hits the ground on which the bowling ball rests.

Modelling the marble as a particle, calculate how far it will land from the base of the bowling ball.

Rolling Projectiles Answers

Part 1

For the marble to leave the sphere, the normal reaction must reach 0.



Resolving radially:

$$mg\cos\theta - R = \frac{mv^2}{r} \quad \Rightarrow \quad \boldsymbol{v^2 = gr\cos\theta}$$

Using conservation of energy:

$$mg(2r) = \frac{1}{2}mv^2 + mgr(1+\cos\theta)$$

$$\Rightarrow \quad \boldsymbol{2gr(1-\cos\theta) = v^2}$$

Equating the two expressions for v^2 gives:

$$gr\cos\theta = 2gr(1-\cos\theta) \quad \Rightarrow \quad 3gr\cos\theta = 2gr \quad \Rightarrow \quad \boldsymbol{\cos\theta = \frac{2}{3}}$$

This gives an angle of $\theta = 48.2°$ to 1 d.p., although the following forms are more use to us:

$$\cos\theta = \frac{2}{3} \quad and \quad \sin\theta = \frac{\sqrt{5}}{3}$$

Note: The expression for $\sin\theta$ is derived from a suitable right-angled triangle by Pythagoras' Theorem.

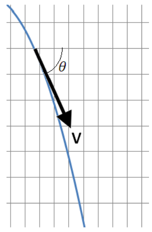
Given a value for $\cos\theta$, v^2 can now be calculated (in terms of r):

$$\boldsymbol{v^2} = gr\cos\theta = \boldsymbol{\frac{2}{3}gr}$$

Note: The alternative expression, $v^2 = 2gr(1-\cos\theta)$, gives the same result.

Rolling Projectiles Answers

Part 2



The angle of velocity at the moment the marble leaves the surface of the bowling ball is the same as the angle from the vertical at which it leaves.

Given the velocity and direction, it should be possible to calculate the time of flight and therefore the range travelled. Then it will be a case of adding on the distance already moved outwards from the being in line with the base of the bowling ball.

Examining horizontal motion:

$$v = \frac{x}{t} \quad \Rightarrow \quad V\cos\theta = \frac{x}{T} \quad \Rightarrow \quad \boldsymbol{x = \frac{2}{3}\left(\sqrt{\frac{2}{3}gr}\right)T}$$

Examining vertical motion:

$$s = r(1+\cos\theta) = \boldsymbol{\frac{5}{3}r} \qquad u = V\sin\theta = \left(\sqrt{\frac{2}{3}gr}\right)\frac{\sqrt{5}}{3} = \frac{1}{3}\sqrt{\boldsymbol{\frac{10}{3}gr}} \qquad a = \boldsymbol{g} \qquad t = \boldsymbol{T}$$

$$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad \frac{5}{3}r = \left(\sqrt{\frac{10}{27}gr}\right)T + \frac{g}{2}T^2 \quad \Rightarrow \quad \frac{g}{2}T^2 + \left(\sqrt{\frac{10}{27}gr}\right)T - \frac{5}{3}r = 0$$

$$T = \frac{-\sqrt{\frac{10}{27}gr} \pm \sqrt{\frac{10}{27}gr + \frac{10}{3}gr}}{g} \quad \Rightarrow \quad T = \frac{-\sqrt{\frac{10}{27}gr} \pm \sqrt{\frac{100gr}{27}}}{g}$$

$$\Rightarrow \quad T = \frac{\sqrt{r}\left(-\sqrt{10} \pm 10\right)}{\sqrt{27g}} \qquad\qquad T > 0 \quad \Rightarrow \quad T = \left(\sqrt{\frac{r}{27g}}\right)\left(10 - \sqrt{10}\right)$$

Using horizontal motion:

$$x = \frac{2}{3}\left(\sqrt{\frac{2}{3}gr}\right)T \quad \Rightarrow \quad x = \frac{2}{3}\left(\sqrt{\frac{2}{3}gr}\right)\left(\sqrt{\frac{r}{27g}}\right)\left(10 - \sqrt{10}\right)$$

$$= \frac{2}{3}\sqrt{\frac{2gr^2}{81g}}\left(10 - \sqrt{10}\right) = \frac{2r}{27}(\sqrt{2}\left(10 - \sqrt{10}\right) = \frac{4}{27}\left(5\sqrt{2} - \sqrt{5}\right)r$$

This is the distance from the start of projectile motion to the end, so we need to include the initial distance of the particle from the centre of the large sphere at the beginning of projectile motion:

$$X = x + r\sin\theta = \frac{4}{27}\left(5\sqrt{2} - \sqrt{5}\right)r + \frac{\sqrt{5}}{3}r$$

$$= \left(\frac{1}{27}\left(20\sqrt{2} - 4\sqrt{5}\right) + \frac{9\sqrt{5}}{27}\right)r = \frac{1}{27}\left(20\sqrt{2} + 5\sqrt{5}\right)r = \boldsymbol{\frac{5}{27}\left(4\sqrt{2} + \sqrt{5}\right)r = 1.46r \; to \; 3 \; s.f.}$$