Practice examination questions for inequalities, asymptotes and further rational functions

1. June 2005

9 The function f is defined by

$$f(x) = \frac{x^2 + 4x}{x^2 + 9}$$

(a) (i) The graph of $y = f(x)$ has an asymptote which is parallel to the x-axis. Find the equation of this asymptote. *(1 mark)*

(ii) Explain why the graph of $y = f(x)$ has no asymptotes parallel to the y-axis. *(2 marks)*

(b) Show that the equation $f(x) = k$ has two equal roots if $9k^2 - 9k - 4 = 0$. *(4 marks)*

(c) Hence find the coordinates of the two stationary points on the graph of $y = f(x)$. *(6 marks)*

2. Jan 2006

4 A curve has equation

$$y = \frac{6x}{x - 1}$$

(a) Write down the equations of the two asymptotes to the curve. *(2 marks)*

(b) Sketch the curve and the two asymptotes. *(4 marks)*

(c) Solve the inequality

$$\frac{6x}{x - 1} < 3$$ *(4 marks)*

3. June 2006

9 A curve C has equation

$$y = \frac{(x + 1)(x - 3)}{x(x - 2)}$$

(a) (i) Write down the coordinates of the points where C intersects the x-axis. *(2 marks)*

(ii) Write down the equations of all the asymptotes of C. *(3 marks)*

(b) (i) Show that, if the line $y = k$ intersects C, then

$$(k - 1)(k - 4) \geqslant 0$$ *(5 marks)*

(ii) Given that there is only one stationary point on C, find the coordinates of this stationary point.

(No credit will be given for solutions based on differentiation.) *(3 marks)*

(c) Sketch the curve C. *(3 marks)*

4. Jan 2007

5 A curve has equation

$$y = \frac{x}{x^2 - 1}$$

(a) Write down the equations of the three asymptotes to the curve. *(3 marks)*

(b) Sketch the curve.

(You are given that the curve has no stationary points.) *(4 marks)*

(c) Solve the inequality

$$\frac{x}{x^2 - 1} > 0$$ *(3 marks)*

5. June 2007

7 A curve has equation

$$y = \frac{3x - 1}{x + 2}$$

(a) Write down the equations of the two asymptotes to the curve. *(2 marks)*

(b) Sketch the curve, indicating the coordinates of the points where the curve intersects the coordinate axes. *(5 marks)*

(c) Hence, or otherwise, solve the inequality

$$0 < \frac{3x - 1}{x + 2} < 3$$ *(2 marks)*

6. June 2008

7 A curve C has equation

$$y = 7 + \frac{1}{x + 1}$$

(a) Define the translation which transforms the curve with equation $y = \frac{1}{x}$ onto the curve C. *(2 marks)*

(b) (i) Write down the equations of the two asymptotes of C. *(2 marks)*

(ii) Find the coordinates of the points where the curve C intersects the coordinate axes. *(3 marks)*

(c) Sketch the curve C and its two asymptotes. *(3 marks)*

7. Jan 2009

6 A curve has equation

$$y = \frac{(x-1)(x-3)}{x(x-2)}$$

(a) (i) Write down the equations of the three asymptotes of this curve. *(3 marks)*

(ii) State the coordinates of the points at which the curve intersects the x-axis. *(1 mark)*

(iii) Sketch the curve.

(You are given that the curve has no stationary points.) *(4 marks)*

(b) Hence, or otherwise, solve the inequality

$$\frac{(x-1)(x-3)}{x(x-2)} < 0$$ *(2 marks)*

8. June 2009

8 A curve has equation

$$y = \frac{x^2}{(x-1)(x-5)}$$

(a) Write down the equations of the three asymptotes to the curve. *(3 marks)*

(b) Show that the curve has no point of intersection with the line $y = -1$. *(3 marks)*

(c) (i) Show that, if the curve intersects the line $y = k$, then the x-coordinates of the points of intersection must satisfy the equation

$$(k-1)x^2 - 6kx + 5k = 0$$ *(2 marks)*

(ii) Show that, if this equation has equal roots, then

$$k(4k+5) = 0$$ *(2 marks)*

(d) Hence find the coordinates of the two stationary points on the curve. *(5 marks)*

9. June 2010

7 (a) (i) Write down the equations of the two asymptotes of the curve $y = \frac{1}{x-3}$. *(2 marks)*

(ii) Sketch the curve $y = \frac{1}{x-3}$, showing the coordinates of any points of intersection with the coordinate axes. *(2 marks)*

(iii) On the same axes, again showing the coordinates of any points of intersection with the coordinate axes, sketch the line $y = 2x - 5$. *(1 mark)*

(b) (i) Solve the equation

$$\frac{1}{x-3} = 2x - 5$$ *(3 marks)*

(ii) Find the solution of the inequality

$$\frac{1}{x-3} < 2x - 5$$ *(2 marks)*

10.Jan 2011

7 A graph has equation

$$y = \frac{x-4}{x^2+9}$$

(a) Explain why the graph has no vertical asymptote and give the equation of the horizontal asymptote. *(2 marks)*

(b) Show that, if the line $y = k$ intersects the graph, the x-coordinates of the points of intersection of the line with the graph must satisfy the equation

$$kx^2 - x + (9k + 4) = 0$$ *(2 marks)*

(c) Show that this equation has real roots if $-\frac{1}{2} \leqslant k \leqslant \frac{1}{18}$. *(5 marks)*

(d) Hence find the coordinates of the two stationary points on the curve.

(No credit will be given for methods involving differentiation.) *(6 marks)*

11.June 2011

8 A curve has equation $y = \dfrac{1}{x^2 - 4}$.

(a) (i) Write down the equations of the three asymptotes of the curve. *(3 marks)*

(ii) Sketch the curve, showing the coordinates of any points of intersection with the coordinate axes. *(4 marks)*

(b) Hence, or otherwise, solve the inequality

$$\frac{1}{x^2 - 4} < -2$$ *(3 marks)*

12.Jan 2012

7 A hyperbola H has equation

$$\frac{x^2}{9} - y^2 = 1$$

(a) Find the equations of the asymptotes of H. *(1 mark)*

(b) The asymptotes of H are shown in the diagram opposite. On the same diagram, sketch the hyperbola H. Indicate on your sketch the coordinates of the points of intersection of H with the coordinate axes. *(3 marks)*

(c) The hyperbola H is now translated by the vector $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

(i) Write down the equation of the translated curve. *(2 marks)*

(ii) Calculate the coordinates of the two points of intersection of the translated curve with the line $y = x$. *(4 marks)*

(d) From your answers to part **(c)(ii)**, **deduce** the coordinates of the points of intersection of the original hyperbola H with the line $y = x - 3$. *(2 marks)*

13.June 2012

5 The curve C has equation $y = \dfrac{x}{(x+1)(x-2)}$.

The line L has equation $y = -\frac{1}{2}$.

(a) Write down the equations of the asymptotes of C. *(3 marks)*

(b) The line L intersects the curve C at two points. Find the x-coordinates of these two points. *(2 marks)*

(c) Sketch C and L on the same axes.

(You are given that the curve C has no stationary points.) *(3 marks)*

(d) Solve the inequality

$$\frac{x}{(x+1)(x-2)} \leqslant -\frac{1}{2}$$ *(3 marks)*

Answer

1. June 2005

9(a)(i)	Asymptote is $y = 1$	B1	1	
(ii)	Denominator never zero	E2,1	2	E1 if incomplete
(b)	$f(x) = k \Rightarrow (1-k)x^2 + 4x - 9k = 0$	M1A1		
	Equal roots if $16 + 36k(1-k) = 0$	m1		
	ie if $9k^2 - 9k - 4 = 0$	A1	4	convincingly shown (AG)
(c)	Solving quadratic for k	M1		
	$k = -\frac{1}{3}$ or $k = \frac{4}{3}$	A1		NMS 2/2
	$4x^2+12x+9 = 0$ or $x^2-12x+36 = 0$	m1		
	SPs when $x = -\frac{3}{2}$ …. ;and when $x =$	A1		
	6…….	A1		
	SPs are $\left(-\frac{3}{2}, -\frac{1}{3}\right)$ and $\left(6, \frac{4}{3}\right)$	A1	6	
	Total		**13**	

2. Jan 2006

4(a)	Asymptotes $x = 1$, $y = 6$	B1B1	2	
(b)	Curve (correct general shape)	M1		SC Only one branch:
	Curve passing through origin	A1		B1 for origin
	Both branches approaching $x = 1$	A1		B1 for approaching both asymptotes
	Both branches approaching $y = 6$	A1	4	(Max 2/4)
(c)	Correct method	M1		
	Critical values ± 1	B1B1		From graph or calculation
	Solution set $-1 < x < 1$	A1√	4	ft one error in CVs; NMS 4/4 after a good graph
	Total		**10**	

3. June 2006

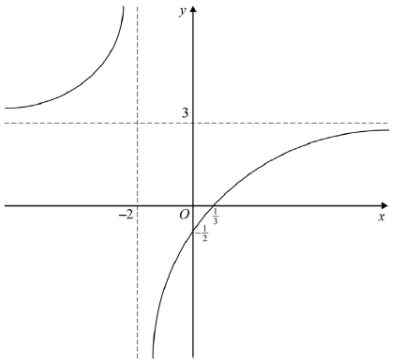
9(a)(i)	Intersections at $(-1, 0)$, $(3, 0)$	B1B1	2	Allow $x = -1$, $x = 3$
(ii)	Asymptotes $x = 0$, $x = 2$, $y = 1$	B1 × 3	3	
(b)(i)	$y = k \Rightarrow kx^2 - 2kx = x^2 - 2x - 3$	M1A1		M1 for clearing denominator
	… $\Rightarrow (k-1)x^2+(-2k+2)x+3=0$	A1√		ft numerical error
	$\Delta = 4(k-1)(k-4)$, hence result	m1A1	5	convincingly shown (AG)
(ii)	$y = 4$ at SP	B1		
	$3x^2 - 6x + 3 = 0$, so $x = 1$	M1A1	3	A0 if other point(s) given
(c)	Curve with three branches	B1		approaching vertical asymptotes
	Middle branch correct	B1		Coordinates of SP not needed
	Other two branches correct	B1	3	3 asymptotes shown
	Total		**16**	

4. Jan 2007

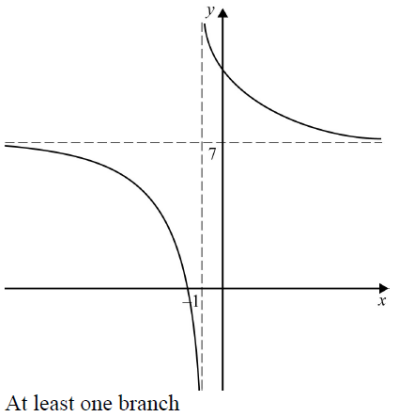
5(a)	Asymptotes $y = 0$, $x = -1$, $x = 1$	B1 × 3	3	
(b)	Three branches approaching two vertical asymptotes	B1		Asymptotes not necessarily drawn
	Middle branch passing through O	B1		with no stationary points
	Curve approaching $y = 0$ as $x \to \pm\infty$	B1		
	All correct	B1	4	with asymptotes shown and curve approaching all asymptotes correctly

(c)	Critical values $x = -1$, 0 and 1	B1		
	Solution set $-1 < x < 0$, $x > 1$	M1A1	3	M1 if one part correct or consistent with c's graph
	Total		**10**	

5. June 2007

7(a)	Asymptotes $x = -2$, $y = 3$	B1,B1	2	
(b)		B1		Curve approaching asymptotes
		B1,B1		Passing through $\left(\frac{1}{3}, 0\right)$ and $\left(0, -\frac{1}{2}\right)$
		B1,B1	5	Both branches generally correct B1 if two branches shown
(c)	Solution set is $x > \frac{1}{3}$	B2,1F	2	B1 for good attempt; ft wrong point of intersection
	Total		**9**	

6. June 2008

7(a)	Curve translated 7 in y direction	B1		
	... and 1 in negative x direction	B1	2	or answer in vector form
(b)(i)	Asymptotes $x = -1$ and $y = 7$	B1B1	2	
(ii)	Intersections at (0, 8) ...	B1		
	... and $(-8/7, 0)$	M1A1	3	Allow AWRT –1.14; NMS 1/2
(c)				
	At least one branch	B1		of correct shape
	Complete graph	B1		translation of $y = 1/x$
	All correct including asymptotes	B1	3	in roughly correct positions
	Total		**10**	

7. Jan 2009

6(a)(i)	Asymptotes $x = 0$, $x = 2$, $y = 1$	B1×3	3	
(ii)	Intersections at (1, 0) and (3, 0)	B1	1	
(iii)	At least one branch approaching asymptotes	B1		
	Each branch	B1×3	4	
(b)	$0 < x < 1$, $2 < x < 3$	B1,B1	2	Allow B1 if one repeated error occurs, eg ≤ for <
	Alternative: Complete correct algebraic method	M1A1	(2)	
	Total		**10**	

8. June 2009

8(a)	Asymptotes $x = 1$, $x = 5$, $y = 1$	B1 × 3	3	
(b)	$y = -1 \Rightarrow (x-1)(x-5) = -x^2$	M1		OE
	$\ldots \Rightarrow 2x^2 - 6x + 5 = 0$	m1		OE
	Disc't = 36 − 40 < 0, so no pt of int'n	A1	3	convincingly shown (AG)
(c)(i)	$y = k \Rightarrow x^2 = k(x^2 - 6x + 5)$	M1		OE
	$\ldots \Rightarrow (k-1)x^2 - 6kx + 5k = 0$	A1	2	convincingly shown (AG)
(ii)	Discriminant = $36k^2 - 20k(k-1)$	M1		OE
	... = 0 when $k(4k + 5) = 0$	A1	2	convincingly shown (AG)
(d)	$k = 0$ gives $x = 0$, $y = 0$	B1		
	$k = -\frac{5}{4}$ gives $-\frac{9}{4}x^2 + \frac{30}{4}x - \frac{25}{4} = 0$	M1A1		OE
	$(3x-5)^2 = 0$, so $x = \frac{5}{3}$	A1		
	$y = -\frac{5}{4}$	B1	5	
	Total		**15**	

9. June 2010

7(a)(i)	Asymptotes $x = 3$ and $y = 0$	B1,B1	2	may appear on graph
(ii)	Complete graph with correct shape	B1		
	Coordinates $\left(0, -\frac{1}{3}\right)$ shown	B1	2	
(iii)	Correct line, (0, −5) and (2.5, 0) shown	B1	1	
(b)(i)	$2x^2 - 11x + 14 = 0$	B1		
	$x = 2$ or $x = 3.5$	M1A1	3	M1 for valid method for quadratic
(ii)	$2 < x < 3$, $x > 3.5$	B2,1F	2	B1 for partially correct solution; ft incorrect roots of quadratic (one above 3, one below 3)
	Total		**10**	

10.Jan 2011

7(a)	Denom never zero, so no vert asymp	E1		
	Horizontal asymptote is $y = 0$	B1	2	
(b)	$x - 4 = k(x^2 + 9)$	M1		
	Hence result clearly shown	A1	2	AG
(c)	Real roots if $b^2 - 4ac \geq 0$	E1		PI (at any stage)
	Discriminant $= 1 - 4k(9k + 4)$	M1		
	$\ldots = -(36k^2 + 16k - 1)$	m1		m1 for expansion
	$\ldots = -(18k - 1)(2k + 1)$	m1		m1 for correct factorisation
	Result (AG) clearly justified	A1	5	eg by sketch or sign diagram
(d)	$k = -\frac{1}{2} \Rightarrow -\frac{1}{2}x^2 - x - \frac{1}{2} = 0$	M1A1		or equivalent using $k = \frac{1}{18}$
	$\ldots \Rightarrow (x+1)^2 = 0 \Rightarrow x = -1$	A1		
	$k = \frac{1}{18} \Rightarrow \frac{1}{18}x^2 - x + \frac{9}{2} = 0$	A1		
	$\ldots \Rightarrow (x-9)^2 = 0 \Rightarrow x = 9$	A1		
	SPs are $\left(-1, -\frac{1}{2}\right), \left(9, \frac{1}{18}\right)$	A1	6	correctly paired
	Total		**15**	

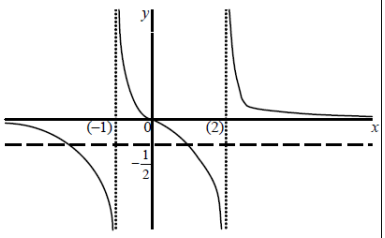
11.June 2011

8(a)(i)	Asymptotes $x = -2$, $x = 2$, $y = 0$	B1 × 3	3	
(ii)	Middle branch generally correct	B1		Allow if max pt not in right place
	Other branches generally correct	B1		
	All branches approaching asymps	B1		Asymps must be shown correctly
	Intersection at $\left(0, -\frac{1}{4}\right)$ indicated	B1	4	on diagram or elsewhere; B0 if any other intersections are shown
(b)	$y = -2$ when $x = \pm\sqrt{3.5}$	B1		Allow NMS
	Sol'n $-2 < x < -\sqrt{3.5}$, $\sqrt{3.5} < x < 2$	B2,1	3	Condone dec approx'n for $\sqrt{3.5}$; B1 if $\leq$ used instead of $<$
	Total		**10**	

12 June 2011

8(a)(i)	Asymptotes $x = -2$, $x = 2$, $y = 0$	B1 × 3	3	
(ii)	Middle branch generally correct	B1		Allow if max pt not in right place
	Other branches generally correct	B1		
	All branches approaching asymps	B1		Asymps must be shown correctly
	Intersection at $\left(0, -\frac{1}{4}\right)$ indicated	B1	4	on diagram or elsewhere; B0 if any other intersections are shown
(b)	$y = -2$ when $x = \pm\sqrt{3.5}$	B1		Allow NMS
	Sol'n $-2 < x < -\sqrt{3.5}$, $\sqrt{3.5} < x < 2$	B2,1	3	Condone dec approx'n for $\sqrt{3.5}$; B1 if $\leq$ used instead of $<$
	Total		**10**	

13 June 2012

5(a)	Asymptotes $x = -1$ $x = 2$ $y = 0$	 B1 B1 B1	 3	 $x = -1$ OE $x = 2$ OE $y = 0$
(b)	$-\frac{1}{2} = \frac{x}{x^2 - x - 2} \Rightarrow x^2 - x - 2 = -2x$ $x^2 + x - 2 = 0 \Rightarrow x = 1,\ x = -2$	M1 A1	 2	Correctly removing brackets and fractions to reach $x^2 - x - 2 = -2x$ OE Correct two values for x-coordinates. NMS 2 or 0 marks
(c)		M1 A1 B1	 3	Three branches shown on sketch of C with either middle branch or outer two branches correct in shape All three branches, correct shape and positions and approaching correct asymptotes in a correct manner. If middle branch does **clearly** not go through the origin, then A0 Correct sketch of line (L), $y = -0.5$ identified
(d)	$-2 \le x < -1$ $1 \le x < 2$ $-2 \le x < -1,\ 1 \le x < 2$	B1 B1 B1	 3	Condone $<$ for $\le$ or vice versa Condone $<$ for $\le$ or vice versa All complete and correct
	Total		**11**	