Pedal Power

The humble bicycle is an incredibly energy-efficient piece of machinery.

Used all over the world for moving people and cargo, they are the best solution we have for a self-powered means of transport.

1. Wikipedia claims that a cyclist with a power output of $60\, \text{W}$ (the same power as someone walking at $5\, \text{kmph}$ ($1.4\, \text{ms}^{-1}$)) can travel at around $15\, \text{kmph}$ ($4.2\, \text{ms}^{-1}$). Assuming that the cyclist is travelling along a straight, horizontal road, and that the resistance forces acting are proportional to its speed, find an expression for the total resistive force acting on the cyclist at a speed $v$.

2. It is estimated that an amateur cyclist can typically sustain $3\, \text{W}$ of power per $\text{kg}$. For example, a $70\, \text{kg}$ rider could output around $210\, \text{W}$ for an extended period of time. Calculate the speed of a $70\, \text{kg}$ rider making this power output.

3. A professional can sustain $6\, \text{W}$ per $\text{kg}$. What would the speed be for our $70\, \text{kg}$ rider in this case?

4. For brief periods, professional cyclists can increase their power output to $25\, \text{W}$ per $\text{kg}$. What would the top speed of a $70\, \text{kg}$ cyclist be while generating this much power?

5. In reality, resistance forces are much more accurately modelled as proportional to the square of the speed. Answer questions 1 to 4 again, using this refined model.

*6. Our amateur cyclist (who can output $210\, \text{W}$ of power and weighs $70\, \text{kg}$) is now cycling up a steep hill, inclined at $5^\circ$ to the horizontal (this is a grade of around 10%). Construct a force diagram, and use it to find his maximum speed up the hill.
Pedal Power SOLUTIONS

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1. Wikipedia claims that a cyclist with a power output of 60W (the same power as someone walking at 5 kmph (1.4 ms\(^{-1}\)) can travel at around 15 kmph (4.2 ms\(^{-1}\)). Assuming that the cyclist is travelling along a straight, horizontal road, and that the resistance forces acting are proportional to its speed, find an expression for the total resistive force acting on the cyclist at a speed v.

\[ P = F_m v \quad \Rightarrow \quad 60 = 4.2F_m \quad \Rightarrow \quad F_m = 14.28 \ldots N \quad \text{and} \quad F_R = kv = 4.2k \\
\]

\[ a = 0 \quad \Rightarrow \quad F_R = F_m \quad \Rightarrow \quad 4.2k = \frac{60}{4.2} \quad \Rightarrow \quad k = \frac{60}{4.2^2} = 3.40 \text{ to 3 s.f.} \\
F_R = 3.40v \]

2. It is estimated that an amateur cyclist can typically sustain 3W of power per kg. For example, a 70kg rider could output around 210W for an extended period of time. Calculate the speed of a 70kg rider making this power output.

\[ P = F_m v \quad \Rightarrow \quad 210 = F_m v \quad \text{and} \quad F_R = 3.4v \]

Max speed \[ \frac{210}{v} = 3.4v \quad \Rightarrow \quad 210 = 3.4v^2 \quad \Rightarrow \quad v = 7.86 \text{ms}^{-1} \text{ to 3 s.f.} \]

3. A professional can sustain 6W per kg. What would the speed be for our 70kg rider in this case?

\[ \frac{420}{v} = 3.4v \quad \Rightarrow \quad v^2 = \frac{420}{3.4} \quad \Rightarrow \quad v = 11.11 \text{ms}^{-1} \text{ to 3 s.f.} \]

4. For brief periods, professional cyclists can increase their power output to 25W per kg. What would the top speed of a 70kg cyclist be while generating this much power?

\[ P = 25 \times 70 = 1750 = F_m v \quad \text{and} \quad F_R = 3.4v \]

Max speed \[ \frac{1750}{v} = 3.4v \quad \Rightarrow \quad v^2 = \frac{1750}{3.4} \quad \Rightarrow \quad v = 22.7 \text{ms}^{-1} \text{ to 3 s.f.} \]

5. In reality, resistance forces are much more accurately modelled as proportional to the square of the speed. Answer questions 1 to 4 again, using this refined model.

1. \[ P = F_m v \quad \Rightarrow \quad 60 = 4.2F_m \quad \Rightarrow \quad F_m = 14.28 \ldots N \quad \text{and} \quad F_R = k v^2 = 4.2k \\
\]

\[ a = 0 \quad \Rightarrow \quad F_R = F_m \quad \Rightarrow \quad 4.2k = \frac{60}{4.2} \quad \Rightarrow \quad k = \frac{60}{4.2^2} = 0.810 \text{ to 3 s.f.} \\
F_R = 0.810v^2 \]

2. \[ P = F_m v \quad \Rightarrow \quad 210 = F_m v \quad \text{and} \quad F_R = 0.81v^2 \]

Max speed \[ \frac{210}{v} = 0.81v^2 \quad \Rightarrow \quad 210 = 0.81v^3 \quad \Rightarrow \quad v = 6.38 \text{ms}^{-1} \text{ to 3 s.f.} \]

3. \[ \frac{420}{v} = 0.81v^2 \quad \Rightarrow \quad v^3 = \frac{420}{0.81} \quad \Rightarrow \quad v = 8.03 \text{ms}^{-1} \text{ to 3 s.f.} \]

4. \[ P = 25 \times 70 = 1750 = F_m v \quad \text{and} \quad F_R = 0.81v^2 \]

Max speed \[ \frac{1750}{v} = 0.81v^2 \quad \Rightarrow \quad v^3 = \frac{1750}{0.81} \quad \Rightarrow \quad v = 12.9 \text{ms}^{-1} \text{ to 3 s.f.} \]
Our amateur cyclist (who can output $210W$ of power and weighs $70kg$) is now cycling up a steep hill, inclined at $5^\circ$ to the horizontal (this is a grade of around 10%). Construct a force diagram, and use it to find his maximum speed up the hill.

\[ P = F_m v \quad \Rightarrow \quad F_m = \frac{210}{v} \]

\[ F_R = 3.4v \]

\[ a = 0 \quad \Rightarrow \quad F = ma = 0 \quad \Rightarrow \quad F_m - F_R - 70g \sin 5 = 0 \]

\[ \frac{210}{v} - 3.4v - 70g \sin 5 = 0 \]

\[ 210 - 3.4v^2 - (70g \sin 5)v = 0 \]

\[ \Rightarrow \quad v = 3.004... \quad or \quad v = -20.5854... \]

$v = -20.6 \text{ ms}^{-1}$ represents the maximum downslope speed

$v = 3.00 \text{ ms}^{-1}$ to 3 s.f.

Note: changing the force diagram for downward motion results in a very similar quadratic:

\[ 210 - 3.4v^2 + (70g \sin 5)v = 0 \]

Notice that equations of this form have related roots, because:

\[ \frac{-(-b) \pm \sqrt{(-b)^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Which means that instead of 3 and $-20.6$, we would get $-3$ and $20.6$ as our solutions. The algebraic solutions effectively allow for the possibility of a negative motive force and a negative resistive force, yielding a negative ‘maximum’ speed. Making these two forces negative effectively changes the direction of motion, so the results reflect a choice of ‘positive’, and hence are applicable, with an appropriate sign change, to motion either up or down the slope.