AQA

ASSESSMENT and QUALIFICATIONS ALLIANCE

Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Core 3

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for C3

Everything you need to know for Core 3 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

Contents

Chapter 1 – Functions

A **function** is a mapping which can be either **one-to-one** or **many-to-one**.

Eg: $f(x) = 3x - 4$ is a one-to-one function, $f(x) = x^2 + 5$ is a many-to-one function.

The **domain** of a function is the set of **input values** it can take.
The **range** of a function is the set of **output values** it can generate.

Eg: $f(x) = \frac{5}{x^2}$ has the domain $x \neq 0$ and the range $f(x) > 0$.

A function is fully defined by both a **rule** and a **domain.**

Eg: $f(x) = \frac{2x}{(x-3)} \;\; for\ x \neq 3$.

The composite function $f(g(x))$ may be written as $fg(x)$.
It applies the function g to the input x, then applies the function f to the input $g(x)$.

Eg: for $f(x) = x^2 + 3$ and $g(x) = 4x$, the composite function is $fg(x) = (4x)^2 + 3 = 16x^2 + 3$.

The inverse function $f^{-1}(x)$ is such that $ff^{-1}(x) = f^{-1}f(x) = x$.
For a function to have a valid inverse, it must be a one-to-one mapping.

Eg: $f(x) = 3x^2 + 5$

1) Write as $y = 3x^2 + 5$
2) Rearrange to make x the subject: $x = \sqrt{\frac{y-5}{3}}$
3) Substitute x for y and y for x: $y = \sqrt{\frac{x-5}{3}}$
4) Write as function: $f^{-1}(x) = \sqrt{\frac{x-5}{3}}$

Note: In terms of the graphical representation of the function, the inverse is a reflection of the original function in the line $y = x$.

Chapter 2 – Transformations of graphs and the modulus function

The transformation $y = f(x - a) + b$ is a **translation** of $\begin{bmatrix} a \\ b \end{bmatrix}$.

Eg: The function $y = (x + 4)^3 + 7$ is a translation of 4 to the left and 7 up from $y = x^3$.

The transformation $y = kf(x)$ is a **stretch by a factor of k** in the **y direction.**
The transformation $y = f\left(\frac{x}{c}\right)$ is a **stretch by a factor of c** in the **x direction.**

Eg: The function $\mathrm{y} = 3\sin 4x$ is a stretch of factor 3 in the y direction and a stretch of factor ¼ in the x direction from $y = \sin x$.

A transformation of $y = -f(x)$ is a **reflection** in the **x-axis.**
A transformation of $y = f(-x)$ is a **reflection** in the **y-axis**.

Eg: The function $y = -x^2$ is a reflection of the function $y = x^2$ in the x-axis.
The function $y = (-x)^3 + (-x)$ is a reflection of the function $y = x^3 + x$ in the y-axis.

Note: the x-axis reflection changes the sign of the output values, while the y-axis reflection changes the sign of the input values.

The **modulus function** gives the **absolute value of its input**, so that:
$|f(x)| = f(x) \ \ when \ \ f(x) \geq 0$
$|f(x)| = -f(x) \ \ when \ \ f(x) < 0$

Eg: $|5x - 10| = 5x - 10 \ \ when \ \ 5x - 10 \geq 0 \ \ ie \ \ x \geq 2$
$|5x - 10| = -(5x - 10) = 10 - 5x \ \ when \ \ 5x - 10 < 0 \ \ ie \ \ x < 2$

Note: To solve simple equations involving the modulus function, deal with the separate cases (function negative, function non-negative) individually, then discard any solutions not within the valid range. To solve more complex equations or inequalities, sketch a graph.

Chapter 3 – Inverse trigonometric functions and sec, cosec and cot

$\sin^{-1} x$ has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$
$\cos^{-1} x$ has domain $-1 \leq x \leq 1$ and range $0 \leq \cos^{-1} x \leq \pi$
$\tan^{-1} x$ has domain $x \in \mathbb{R}$ and range $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

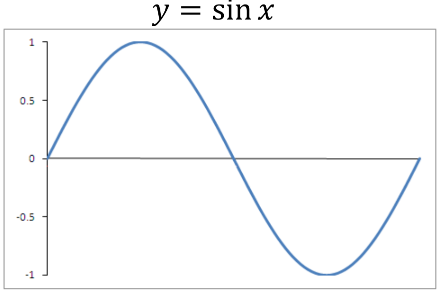
Secant, cosecant and cotangent are defined as: $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\cot x = \frac{1}{\tan x}$.

Note: $\frac{1}{\sin x} \neq \sin^{-1} x$. One is the **reciprocal** of the function, the other is the **inverse** function.

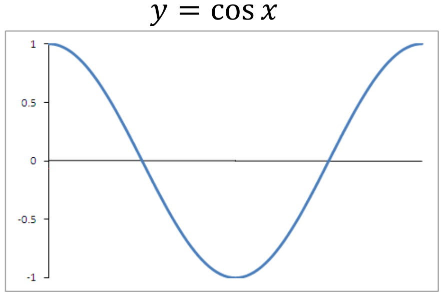
$$\sin^2 x + \cos^2 x \equiv 1$$
$$1 + \cot^2 x \equiv \csc^2 x$$
$$\tan^2 x + 1 \equiv \sec^2 x$$

Note: The second two identities can both be produced easily from the first one by dividing through by either $\sin^2 x$ or $\cos^2 x$.

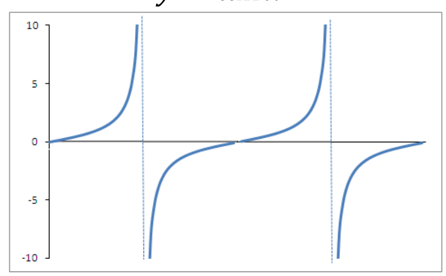
$y = \sin x$



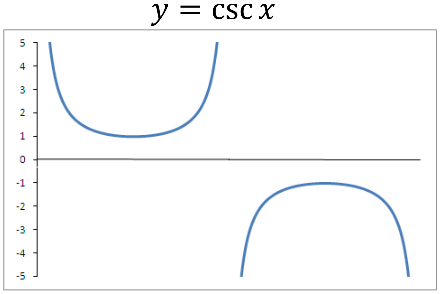
$y = \cos x$



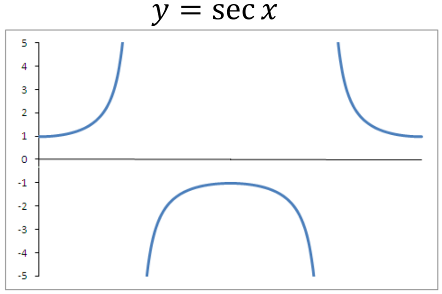
$y = \tan x$



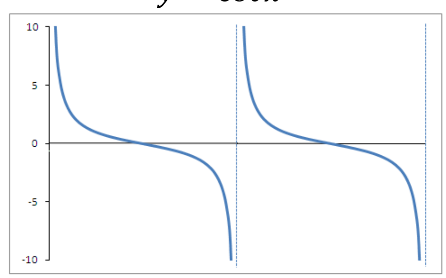
$y = \csc x$



$y = \sec x$



$y = \cot x$



(All horizontal scales between 0^c and $2\pi^c$)

Note: You should know how to accurately sketch the primary trigonometric functions – the other three you ought to be able to produce from those.

Chapter 4 – The number e and calculus

$$x = e^y \Rightarrow y = \ln x$$
$$e^0 = 1 \;\; and \;\; \ln 1 = 0$$

Note: This result can be obtained by taking natural logarithms of both sides and applying the log rules (included below).

$$\ln a + \ln b = \ln ab$$
$$\ln a - \ln b = \ln \tfrac{a}{b}$$
$$n \ln a = \ln a^n$$

$$\frac{d}{dx}(e^x) = e^x \quad and \quad \int e^x \, dx = e^x + C$$
$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b} \quad and \quad \int e^{ax+b} \, dx = \tfrac{1}{a}e^{ax+b} + C$$
$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)} \quad and \quad \int e^{f(x)} \, dx = \tfrac{1}{f'(x)}e^{f(x)} + C$$

(The latter results above are introduced later in the course, but are included here for completeness)

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad and \quad \int \frac{1}{x} \, dx = \ln x + C$$

Note: The integral of $\ln x$ is not readily apparent, but can be achieved using Parts.

Equations involving e^x can often be solved by rearranging to find e^x then taking logs.

Eg: $e^x + 3e^{-x} = 4$

$$e^{2x} + 3 = 4e^x$$

$$e^{2x} - 4e^x + 3 = 0$$

$$(e^x - 1)(e^x - 3) = 0$$

$$e^x = 1 \;\; or \;\; e^x = 3$$

$$x = \ln 1 = 0 \;\; or \;\; x = \ln 3$$

Chapter 5 – Further differentiation and the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

This is the chain rule proper. However, it is often easier to use this expression of it:

$$\frac{d}{dx}fg(x) = f'\big(g(x)\big)g'(x)$$

Eg: $\frac{d}{dx}(2x+5)^3 = 3(2x+5)^2(2) = 6(2x+5)^2$

The $2x + 5$ part is 'ignored', then the end result is multiplied by the differential of the 'ignored' part.

Note: The chain rule can be extended to any number of functions within functions, but you will rarely be expected to deal with more than two or three.

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

Note: A number of more complex trigonometric (and other) results are given in the formula book.

An **increasing function** is a function $f(x)$ such that $f'(x) \geq 0$ for all x.
A **decreasing function** is a function $f(x)$ such that $f'(x) \leq 0$ for all x.

Note: This means that a function is only **increasing** if its gradient is **never negative**, and a function is only **decreasing** if its gradient is **never positive**.

Eg: $f(x) = x^3 + 2x$ is an increasing function because $f'(x) = 3x^2 + 2 > 0$ for all x (since $x^2 \geq 0$).

Chapter 6 – Differentiation using the product and quotient rule

Product Rule:
$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

This is the function notation form. A more easily memorable one may be written as:

$$\frac{d}{dx}uv = uv' + vu'$$

Eg: $\frac{d}{dx}(x^2 \sin x) = x^2 \cos x + 2x \sin x$

Quotient Rule:
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

This is given (in function notation) in the formula book, but this form may be more memorable.

Eg: $\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

Chapter 7 – Numerical solutions of equations and iterative methods

To show that $f(x) = 0$ has a root α such that $a < \alpha < b$, it is sufficient to demonstrate that either $f(a) < 0$ *and* $f(b) > 0$ or $f(a) > 0$ *and* $f(b) < 0$.

Note: The converse is not always true. $f(a) > 0$ *and* $f(b) > 0$ does not imply that there isn't a root between a and b.

If you are required to show a function has a root between two values, first rearrange into the form $f(x) = 0$, then apply the rule above.

Iteration requires a formula linking each approximation of x to the next: $x_{n+1} = f(x_n)$.

An iteration $x_{n+1} = f(x_n)$ converges if $|f'(x)| < 1$ for certain values of x.

If the iteration converges to a limit, that limit can be found by setting $x_{n+1} = x_n = L$.

Eg: The equation $x^3 - 4x = 7$ produces the iteration formula $x_{n+1} = \sqrt[3]{7 + 4x_n}$. Starting with 5:

$$x_1 = 5$$
$$x_2 = \sqrt[3]{7 + 4x_1} = \sqrt[3]{7 + 4(5)} = 3$$
$$x_3 = \sqrt[3]{7 + 4x_2} = \sqrt[3]{7 + 4(3)} = 2.6684 \ldots$$

Chapter 8 – Integration by inspection and substitution

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C \ \ for\ n \neq 1$$

$$\int e^{ax+b} \, dx = \tfrac{1}{a} e^{ax+b} + C$$

Note: These results come from using the substitution $u = ax + b$, but can be quoted directly.

$$\int \cos x \ \, dx = \sin x + C$$

$$\int \sin x \ \, dx = -\cos x + C$$

$$\int \sec^2 x \ \, dx = \tan x + C$$

$$\int \cos(ax+b) \ \, dx = \tfrac{1}{\mathrm{a}}\sin(ax+b) + C$$

$$\int \sin(ax+b) \ \, dx = -\tfrac{1}{\mathrm{a}}\cos(ax+b) + C$$

$$\int \sec^2(ax+b) \ \, dx = \tfrac{1}{a}\tan(ax+b) + C$$

Note: A number of more complex trigonometric results are given in the formula book.

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

This general result (which can be verified by differentiating $\ln|f(x)|$ using the chain rule) is often achievable with some slight tweaking.

Eg:

$$\int \frac{x^2}{x^3-5} \, dx = \tfrac{1}{3}\int \frac{3x^2}{x^3-5} \, dx = \tfrac{1}{3}\ln|x^3-5| + C$$

Chapter 9 – Integration by parts and standard integrals

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Note: **This result is given in the formula book –** it is given here for reference only.

You will be most commonly expected to apply Integration by Parts to the following forms of integral:

$$\int x^n \sin mx\ dx$$

$$\int x^n \cos mx\ dx$$

$$\int x^n e^{mx}\,dx$$

$$\int x^n \ln mx\ dx$$

Note: While for most functions the simplest part (usually x^n) becomes u and the other part becomes $\frac{dv}{dx}$, when $\ln mx$ is involved, this should be made u, since it is not readily integrable.

Eg:

$$\int x^2 \ln x\ dx$$

$$u = \ln x \qquad \frac{dv}{dx} = x^2$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

$$\int x^2 \ln x\ dx = \frac{x^3}{3}\ln x - \int \frac{\frac{x^3}{3}}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C$$

Note: The standard integrals in this chapter are all given in the formula book, but it is important to be familiar with using and applying them.

Chapter 10 – Volume of revolution and numerical integration

The volume of a function $y = f(x)$ rotated fully around the x-axis is given by:

$$V = \int_a^b \pi y^2 \, dx$$

Note: To find a volume rotated around the y-axis, rearrange to give $x = g(y)$ and use the rule:

$$V = \int_a^b \pi x^2 \, dy$$

Approximations to integrals can be found using a **numerical integration** method. Three of the most commonly used methods are:

The Trapezium Rule approximates the curve to straight lines and calculates the area of the resultant trapezia.

The Mid-Ordinate Rule takes the midpoint between two ordinates as the height and approximates the area through a series of rectangles.

Simpson's Rule approximates the curve as a series of quadratics, and since this is usually a much closer fit to the actual function, Simpson's Rule usually gives a much better approximation than either of the other two. Valid only when used with an even number of strips.

Note: All of these rules are quoted in the formula book, but you will need to be familiar with them and be able to apply any of them to a function.

Eg: Approximation to $\int_0^2 \sqrt{1+e^x} \, dx$ using Simpson's Rule with five ordinates (four strips):

$$h = \frac{b-a}{n} = \frac{2}{4} = 0.5$$

$x_0 = 0 \quad y_0 = \sqrt{1+e^0} = 1.4142 \ldots$
$x_1 = 0.5 \quad y_1 = \sqrt{1+e^{0.5}} = 1.6274 \ldots$
$x_2 = 1 \quad y_2 = \sqrt{1+e^1} = 1.9282 \ldots$
$x_3 = 1.5 \quad y_3 = \sqrt{1+e^{1.5}} = 2.3413 \ldots$
$x_4 = 2 \quad y_4 = \sqrt{1+e^2} = 2.8963 \ldots$

$$\int_0^2 \sqrt{1+e^x} \, dx \approx \tfrac{0.5}{3}\big(1.4142 \ldots + 2.8963 \ldots + 4(1.6274 \ldots + 2.3413 \ldots) + 2(1.9282 \ldots)\big) = 24.0423 \ldots$$
$$= 24.042 \ to \ 3dp$$

Note: Simpson's Rule with n ordinates usually gives a result accurate to n decimal places.

Produced by A. Clohesy; TheChalkface.net
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