iSum

Find the value of the sum $1 + i + i^2 + i^3 + \cdots + i^n$ for any integer n.

Solution

First, note that $i^2 = -1$ by definition, so $i^3 = i^2(i) = -i$ and $i^4 = -i(i) = -i^2 = -(-1) = 1$. Therefore the series can be written as:

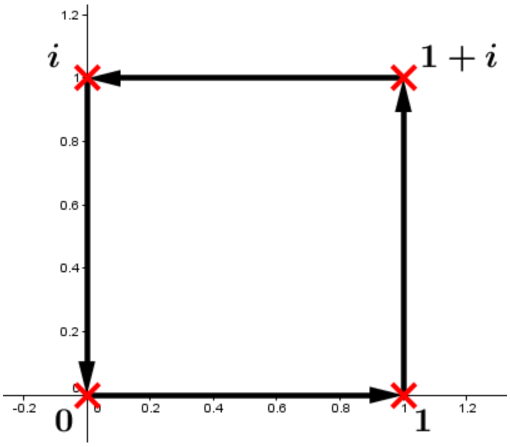
$$1 + i + i^2 + i^3 + \cdots = 1 + i - 1 - i + 1 + i - 1 - i + \cdots$$

The first four terms cancel out, and it is clear from the periodic nature of the sequence that the final term can only be one of four things. Therefore its sum can only take one of four possible values:

When n is 0 (mod 4), the last term will be $i^{4k} = 1$ and the sum will be 1.
When n is 1 (mod 4), the last term will be $i^{4k+1} = i$ and the sum will be $1 + i$.
When n is 2 (mod 4), the last term will be $i^{4k+2} = i^2 = -1$ and the sum will be i.
When n is 3 (mod 4), the last term will be $i^{4k+3} = i^3 = -i$ and the sum will be 0.

Note: $n = 0$ (mod 4) means n is exactly a multiple of 4, while $n = 1$ (mod 4) means n is 1 greater than a multiple of 4, etc.

In terms of vectors on the Argand diagram, adding 1 is equivalent to moving one unit to the right, adding i is equivalent to moving one unit up, and so on. Since the sequence does one after the other, we are simply moving right, up, left, down, right, up, left, down, etc, one unit in each direction. Therefore we are tracing out a square between the points on the Argand diagram represented by 0, i, $1 + i$ and 1:



Extension:
While the above results give us a workable rule, it is somewhat messy to have a rule which must be defined differently for different conditions. Consider the simpler sequence:

$$1 - 1 + 1 - 1 + 1 - \cdots$$

Each term in this sequence is either 1 or -1, and we could define the sum to the k^{th} term as:

When k is 0 (mod 2), the last term will be -1 and the sum will be 0.
When k is 1 (mod 2), the last term will be 1 and the sum will be 1.

However, note that $(-1)^k$ will be 1 whenever k is even and -1 whenever k is odd. By subtracting this from 1 we generate 0 for even k and 2 for odd k, and all that remains is to divide by 2 to give the required results. So our rule for the sum to the k^{th} term can be written as:

$$S_k = \frac{1 - (-1)^k}{2}$$

P.T.O.

For our more complex example (if you'll pardon the pun), it helps to use the concept of geometric series (which you may not have encountered yet). This allows us to find a single formula that describes the sum to the n^{th} power of i:

A geometric series is a sum where each term is a given multiple of the term before (eg $1 + 10 + 100 + \cdots$ or $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$). When the first term is a and the common ratio is r, the sum of the first k terms is:

$$S_k = \frac{a(1 - r^k)}{1 - r}$$

Our series has first term 1 and common ratio i, yielding:

$$S_k = \frac{1(1 - i^k)}{1 - i} = \frac{1 - i^k}{1 - i}$$

Note: For a series up to i^n, as in the question, there will be $n + 1$ terms, so $k = n + 1$. This gives:

$$1 + i + i^2 + \cdots + i^n = \frac{1 - i^{n+1}}{1 - i}$$

Rationalizing the denominator:

$$\frac{(1 - i^{n+1})(1 + i)}{(1 - i)(1 + i)} = \frac{1 - i^{n+1} + i - i(i^{n+1})}{1 - i^2} = \frac{(1 + i - i^{n+1} - i^{n+2})}{2} = \frac{1 + i - i^{n+1} + i^n}{2}$$

$$= \frac{1 + i + (1 - i)i^n}{2}$$

Recall that i^n can equal 1, i, -1 or $-i$ (when n is equal to 0, 1, 2 or 3 respectively).

Checking these four conditions:

$$n = 0 \ (mod\ 4) \quad \Rightarrow \quad \frac{1 + i + (1 - i)}{2} = 1$$

$$n = 1 \ (mod\ 4) \quad \Rightarrow \quad \frac{1 + i + (1 - i)i}{2} = 1 + i$$

$$n = 2 \ (mod\ 4) \quad \Rightarrow \quad \frac{1 + i + (1 - i)(-1)}{2} = i$$

$$n = 3 \ (mod\ 4) \quad \Rightarrow \quad \frac{1 + i + (1 - i)(-i)}{2} = 0$$

Therefore our single formula yields the same results as the original four:

$$\boldsymbol{1 + i + i^2 + \cdots + i^n = \frac{1 + i + (1 - i)i^n}{2}}$$