Introduction to Logarithms

How Your Brain Compares Numbers

Try the following exercises to reveal how your brains tends to deal with comparative size. Try not to over-think these; just go with whichever answer seems the most sensible.

Number-line

On the number-line below, mark on where you think the number 1000 should go:

1 —————————————————————— **1,000,000**

Ball-park guesswork

Four quiz contestants each try to guess the population of Nottingham:

Andy: 6 Becky: 60 Claire: 6,000 Dave: 600,000

Which do you think is the most sensible guess? Which is the least sensible?

Best guess:	
Worst guess:	

The actual population of Nottingham is 300,000. Does this change your choices?

What criteria did you use? Did this give sensible-looking results?

Money matters

You plan to buy a **new car for £8000** and a **new phone for £100** today.

What would you do in each scenario?

A) An identical garage 10 miles further away offers the car for £7980.
Would you travel the extra distance for the £20 saving?

B) An identical phone shop 10 miles further away offers the phone for £80.
Would you travel the extra distance for the £20 saving?

Change the size of the saving – how much of a saving would you need before you'd travel further to buy the car? What about the phone?

Were your answers different if you were saving money on the phone or the car? Why?

A New Sort of Number Line

In some sense, £10,000 is closer to £8,000 than £2,040 is to £40.
This is because we often compare numbers **multiplicatively** rather than **additively**:

$$£10{,}000 \text{ is } \mathbf{1.25} \times £8{,}000, \text{ but } £2{,}040 \text{ is } \mathbf{51} \times £40.$$

This method measures the gap between two numbers as a ***ratio*** rather than the ***difference***.

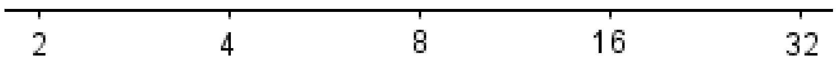
Most number lines you see are *additive*…

Each step adds 1:

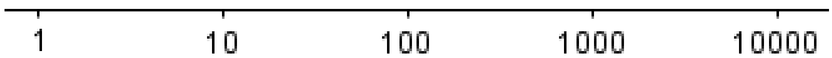
Each step adds 5:

…but we could construct a *multiplicative* scale:

Each step multiplies by 2:



Each step multiplies by 10:



How big are the following numbers in $\times$ 4 counting?

What is the power of 4 that makes 16? $\Leftrightarrow$ *What is* $\log_4 16$*?*

1) 16
2) 64
3) 4
4) 1
5) $\frac{1}{4}$
6) 2
7) $\sqrt{2}$
8) 32
9) 0
10) -4

Clearly using *powers* (aka *exponents*, *indices*, *orders*, *logarithms*) is the most convenient way to represent the new number line in a form that allows us to use what we already know:

$x^a \times x^b = x^{a+b}$	$(x^a)^b = x^{ab}$	$x^1 = x$ and $x^0 = 1$
$\frac{x^a}{x^b} = x^{a-b}$	$x^{-n} = \frac{1}{x^n}$	$x^{\frac{1}{n}} = \sqrt[n]{x}$

The Logarithm Function

The **logarithm** function does the *opposite* of the **exponential** function.

While the exponential function raises a certain base to the given power, the logarithm function starts with the result and reverse-engineers the power needed.

Consider the formula for an exponential function:

$$A = b^x$$

- The number $\boldsymbol{b}$ is called the **base**.
- The number $\boldsymbol{x}$ is called the **logarithm** (aka *power*, *index*, etc).

The function which reverses this (makes x the subject) is called the **log function**:

$$\log_b A = x$$

Since log base b reverses raising b to a power, the two functions can cancel each other out:

$$\log_b(b^x) = x = b^{(\log_b x)}$$

A function that reverses the effect of another is called its *inverse*.
Examples include $x + 3$ and $x - 3$, or $\sin x$ and $\sin^{-1} x$, or x^2 and $\sqrt{x}$.

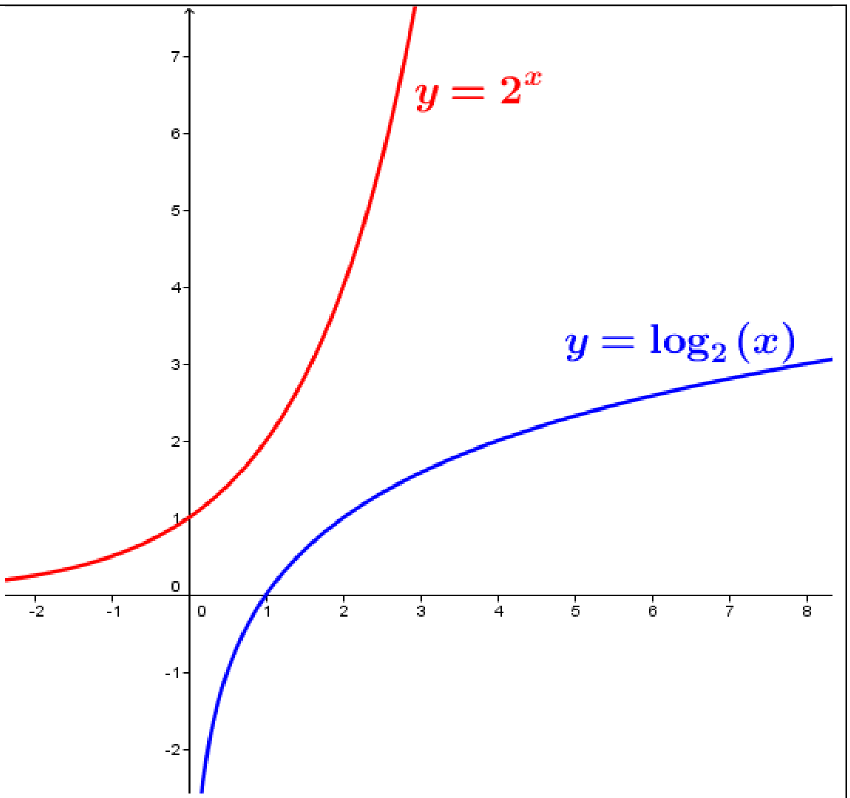
Take care not to confuse x^3 with 3^x. The inverse of x^3 is $\sqrt[3]{x}$, but the inverse of 3^x is $\log_3 x$.

An **exponential function** such as $y = 2^x$ increases, and it does so at an ever-increasing rate.

A **logarithmic function** such as $y = \log_2 x$ also increases, but at an ever-decreasing rate.

Notice that, since these two functions are inverses of one another, each is a reflection of the other in the line $y = x$. This means applying one then the other gets you back where you started:

$$2^5 = 32, \text{ and } \log_2 32 = 5$$



Laws of Logarithms

Since logarithms are indices, the laws are actually the same as the laws of indices, but written from the point of view of the powers (or 'logarithms'):

Index Rule	**Logarithm Rule**	***Underlying Intuition***
$x^a \times x^b = x^{a+b}$	$\log A + \log B = \log AB$	*Multiplying a 5 digit by a 3 digit number gives a (roughly) 8 digit number.*
$\frac{x^a}{x^b} = x^{a-b}$	$\log A - \log B = \log \frac{A}{B}$	*Dividing a 5 digit by a 3 digit number gives a (roughly) 2 digit number.*
$(x^a)^b = x^{ab}$	$\log A^n = n \log A$	*Raising a 5 digit number to the power 3 gives a (roughly) 15 digit number.*
$x^1 = x$ & $x^0 = 1$	$\log 10 = 1$ & $\log 1 = 0$	*Raising a number to the power 1 has no effect, and raising to the power 0 gives 1.*

Note: The x^{-n} and $x^{\frac{1}{n}}$ rules are simply definitions, so do not really constitute 'index laws'.

Logarithms Practice

Recall that $\log_5 125 = ?$ means the same as $5^? = 125$.
Example:

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Answer the questions below (*can you spot the two impossible questions?*)

$\log_4 256 =$ $\qquad$ $\log_2 \frac{1}{16} =$ $\qquad$ $\log_{49} 7 =$ $\qquad$ $\log_9 9 =$

$\log_{17} 1 =$ $\qquad$ $\log_{\frac{1}{2}} 32 =$ $\qquad$ $\log_{16} 0 =$ $\qquad$ $\log_2 -4 =$

Recall that $\log_n p = q$ is equivalent to $n^q = p$.
Example:

$$\log_6 36 = 2 \quad \Longrightarrow \quad 6^2 = 36$$

Rewrite the following statements in exponential form.

$\log_2 32 = 5 \quad \Longrightarrow$ $\qquad$ $\log_{10} 100 = 2 \quad \Longrightarrow$

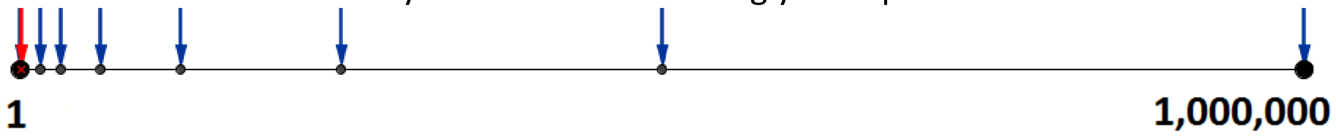
$\log_{0.25} 16 = -2 \quad \Longrightarrow$ $\qquad$ $\log_5 0.2 = -1 \quad \Longrightarrow$

How Your Brain Compares Numbers: The 'solutions'

Although you could calculate many of the answers below 'mathematically', this may lead you to some rather counter-intuitive answers. By going with what *seems* right initially, you hopefully managed to uncover something of the way your brain compares numbers…

Number-line

Which of these arrows was your initial hunch telling you to point?



Most people guess somewhere between a quarter and a third of the way along.
The 'true' answer (for a normal scale at least) is barely distinguishable from 1:
Something in your brain values the gap from 1000 to 1000000 little more than 1 to 1000.
Where do you think you would put a billion on the line from a million to a trillion?

Ball-park guesswork

Four quiz contestants each try to guess the population of Nottingham:

Andy: 6 Becky: 60 Claire: 6,000 Dave: 600,000

Which do you think is the most sensible guess? Which is the least sensible?

	Based on difference	Based on… common sense?
Best guess:	6,000	600,000
Worst guess:	600,000	6

We perceive a bigger gap between 6,000 and 300,000 than between 300,000 and 600,000
The actual population of Nottingham is 300,000. Does this change your choices?
What criteria did you use? Did this give sensible-looking results?
Notice that Dave's guess was 2 times too big, but Claire's was 50 times too small
(Becky's and Andy's guesses were 5,000 and 50,000 times too small respectively)

Money matters

You plan to buy a **new car for £8000** and a **new phone for £100** today.
What would you do in each scenario?

A) An identical garage 10 miles further away offers the car for £7980.
Would you travel the extra distance for the £20 saving?

B) An identical phone shop 10 miles further away offers the phone for £80.
Would you travel the extra distance for the £20 saving?

Change the size of the saving – how much of a saving would you need before you'd travel further to buy the car? What about the phone?

Were your answers different if you were saving money on the phone or the car? Why?
Most people would travel to make a saving on a £100 phone when they wouldn't travel the same distance for the same saving on an £8000 car. We automatically analyse the proportional saving (eg 0.25% on the car or 20% on the phone) rather than the total saving.

A New Sort of Number Line SOLUTIONS

How big are the following numbers in $\times$ 4 counting?

What is the power of 4 that makes 16? $\Leftrightarrow$ *What is log_4 16?*

1) 16	$\log_4 16 = \log_4 4^2 = \mathbf{2}$	6) 2	$\log_4 2 = \log_4 4^{\frac{1}{2}} = \mathbf{\frac{1}{2}}$
2) 64	$\log_4 64 = \log_4 4^3 = \mathbf{3}$	7) $\sqrt{2}$	$\log_4 \sqrt{2} = \log_4 4^{\frac{1}{4}} = \mathbf{\frac{1}{4}}$
3) 4	$\log_4 4 = \log_4 4^1 = \mathbf{1}$	8) 32	$\log_4 32 = \log_4 4^{\frac{5}{2}} = \mathbf{\frac{5}{2}}$
4) 1	$\log_4 1 = \log_4 4^0 = \mathbf{0}$	9) 0	$\log_4 0$ *not possible*
5) $\frac{1}{4}$	$\log_4 \frac{1}{4} = \log_4 4^{-1} = \mathbf{-1}$	10) -4	$\log_4 -4$ *not possible*

Logarithms Practice SOLUTIONS

Recall that $\log_5 125 =?$ means the same as $5^? = 125$.
Example:

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Answer the questions below (*can you spot the two impossible questions?*)

$\log_4 256 = 4$ $\quad$ $\log_2 \frac{1}{16} = -4$ $\quad$ $\log_{49} 7 = \frac{1}{2}$ $\quad$ $\log_9 9 = 1$

$\log_{17} 1 = 0$ $\quad$ $\log_{\frac{1}{2}} 32 = -5$ $\quad$ $\log_{16} 0 =$* $\quad$ $\log_2 -4 =$*

* these two are impossible because no power of 16 can give 0, and no power of a positive number can give a negative answer.

Recall that $\log_n p = q$ is equivalent to $n^q = p$.
Example:

$$\log_6 36 = 2 \quad \Longrightarrow \quad 6^2 = 36$$

Rewrite the following statements in exponential form.

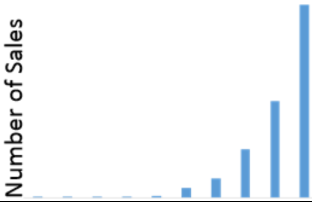
$\log_2 32 = 5 \quad \Longrightarrow \quad 2^5 = 32$ $\qquad$ $\log_{10} 100 = 2 \quad \Longrightarrow \quad 10^2 = 100$

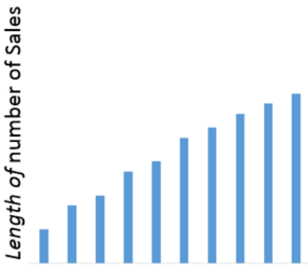
$\log_{0.25} 16 = -2 \quad \Longrightarrow \quad 0.25^{-2} = 16$ $\qquad$ $\log_5 0.2 = -1 \quad \Longrightarrow \quad 5^{-1} = 0.2$

Evidence Of A Multiplicative Number Sense

Yellow Star ( ) = 10 to 49 points
Blue Star ( ) = 50 to 99 points
Turquoise Star ( ) = 100 to 499 points
Purple Star ( ) = 500 to 999 points
Red Star ( ) = 1,000 to 4,999 points
Green Star ( ) = 5,000 to 9,999 points
Yellow Shooting Star ( ) = 10,000 to 24,999 points
Turquoise Shooting Star ( ) = 25,000 to 49,999 points
Purple Shooting Star ( ) = 50,000 to 99,999 points
Red Shooting Star ( ) = 100,000 or more points

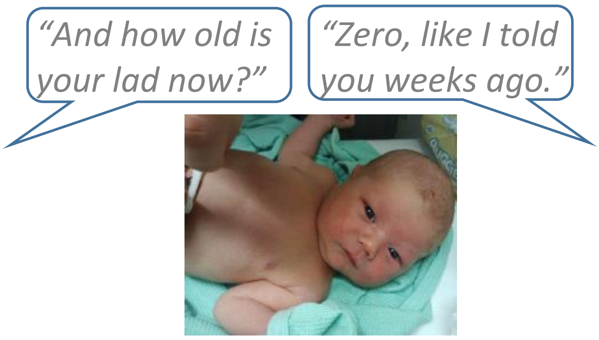
Feedback stars are awarded on a multiplicative scale – count the digits:





Measuring Age

Age (years)	Accepted Units
0 to $\frac{1}{365}$	hours
$\frac{1}{365}$ to $\frac{7}{365}$	days
$\frac{7}{365}$ to $\frac{2}{12}$	weeks
$\frac{2}{12}$ to 1	months
1 to 4	half-years
4 to 20	years
20 to 40	half-decades
40 to 100	decades

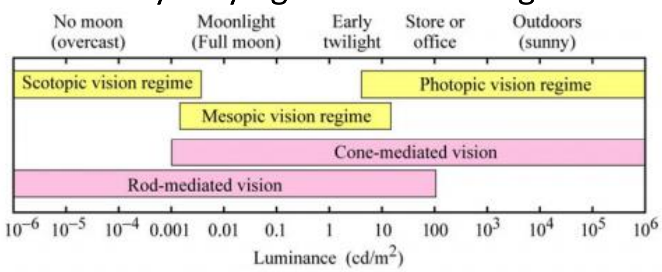


Sound and Light

Our range of effective hearing is massive: we can detect a pin drop or survive a nearby shotgun blast (direction permitting) despite the fact that the sound from the shotgun is 100,000,000,000,000 greater.

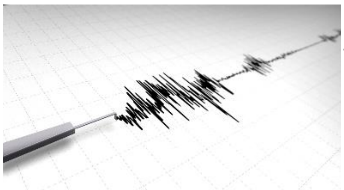
Your ears automatically adjust so that a motorbike doesn't seem 1000 times louder than a vacuum cleaner even though it has 1000 times the sound energy. The Decibel system takes this into account.

Our eyes are similarly flexible when dealing with widely varying intensities of light:



Earthquakes

An earthquake of size 3 on the Richter scale is barely noticeable. A size 4 doesn't have a huge amount more effect, but has around 30 times the energy output.



Magnitude (Richter)	Energy (GigaJoules)	Number (per year)
3	2	20000
4	60	4000
5	2000	600
6	60000	100
7	2000000	20

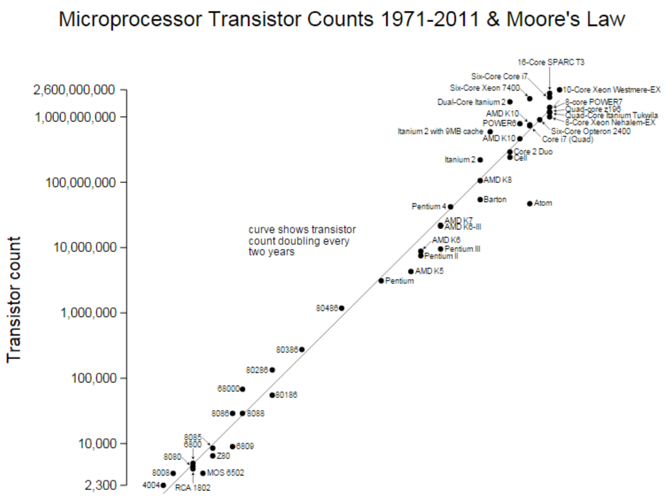
It makes more sense to consider the *number of digits* rather than the numbers themselves when a large change in size only produces a small change in impact.

0-2	Not detectable by people
2-3	Barely detectable by people
3-4	Ceiling lights swing
4-5	Walls crack
5-6	Furniture moves
6-7	Some buildings collapse
7-8	Many buildings collapse
8-9	Total destruction of buildings & roads

Moore's Law

50 years ago, Gordon Moore predicted that computing power* would **double** every two years or so.

This has proved to be such an accurate prediction of computer hardware advances that software companies use it to plan ahead, designing and setting targets for hardware that doesn't even exist yet.



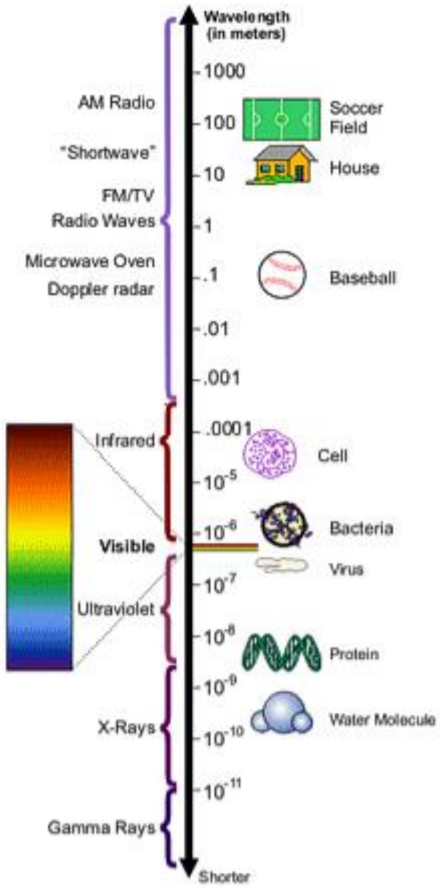
Notice how the vertical scale on this graph is not your average number-line…

The size of memory sticks available for under £30 year on year has been increasing by a similar amount, doubling in size every one or two years:

*Specifically the number of components in a computer chip, but it amounts to the same thing.

Electromagnetic Spectrum

To fully demonstrate the enormous range of the electromagnetic spectrum, from wavelengths the size of car parks down to the size of individual atoms, and everything in between, we need a scale more suited to such enormous changes of size:



Musical Scales

Every time you go up an octave, you *double* the frequency of the note.
Middle A is 440 Hz, high A is 880 Hz and low A is 220 Hz. Below that is 110 Hz, etc.