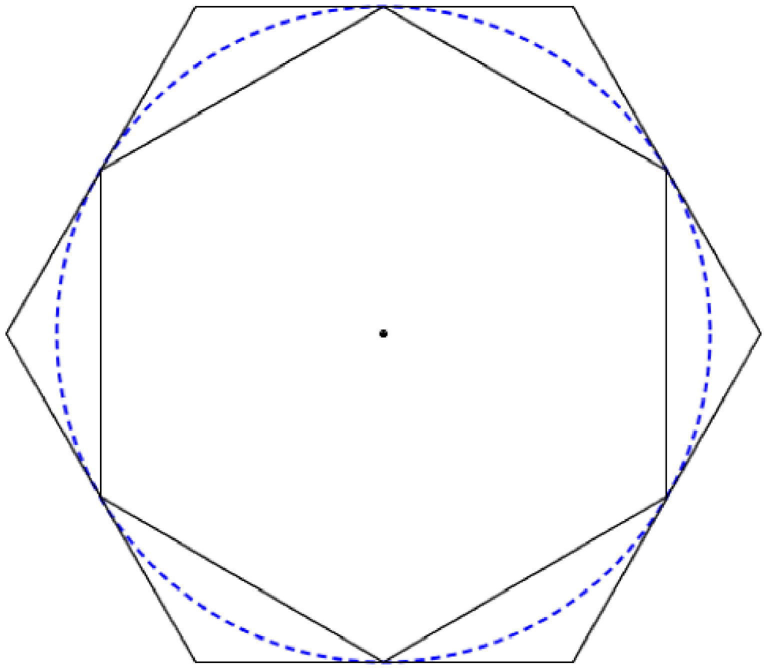
Finding π using trigonometry

The radius of the circle is $1\ unit$.

Find the perimeter of the inscribed (inner) hexagon, and the perimeter of the circumscribed (outer) hexagon.

By taking an average of the two, and comparing to the circumference, $C = 2\pi r$, generate an estimate for π.



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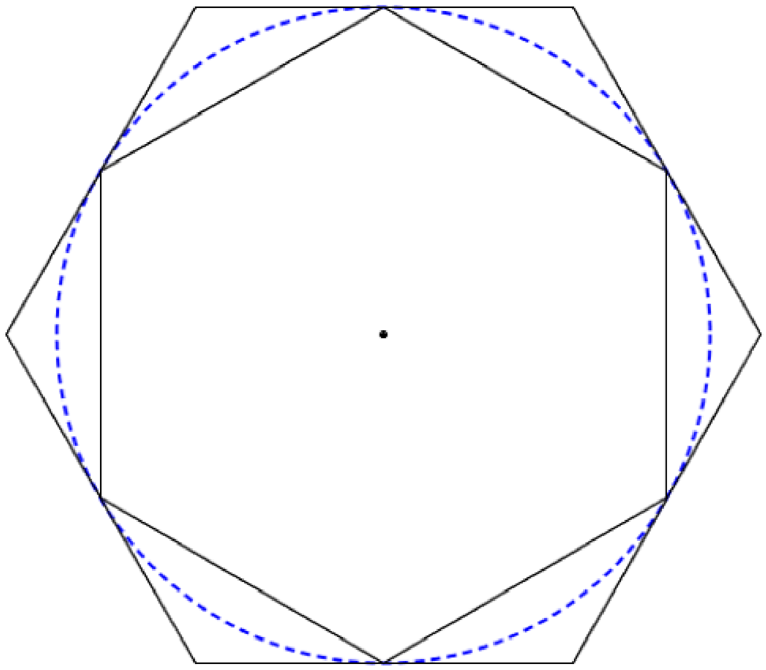

Finding π using trigonometry SOLUTIONS

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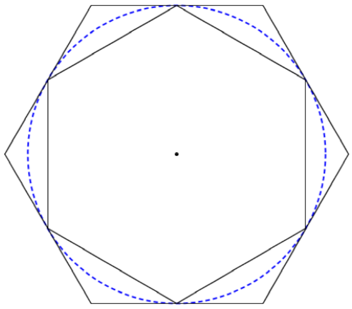

Archimedes used this idea - but with no trigonometry or calculators to help him! - to narrow down the value of π: $3.1408 < \pi < 3.1428$ *He used a doubling technique to build from a hexagon to a 96-gon.*

A hexagon has interior angle $120°$, so constructing lines from the centre to each corner splits the hexagon into six equilateral triangles. Since the distance from the centre to the corner of the **inscribed hexagon** is 1, the length of each edge must also be 1, and the perimeter of the whole shape is therefore $6\ units$.

For the **circumscribed hexagon**, the distance from the centre to the midpoint of an edge is 1. The triangle formed by linking the centre, a midpoint and an adjacent corner has angles $30°$, $60°$ and $90°$ (since it is a bisected equilateral triangle), with the side linking the midpoint with the corner opposite the $30°$ angle. This makes the side of length 1 the adjacent, and the edge we are interested in the opposite. Using the tan ratio we can say $\tan 30 = \frac{x}{1}$. Since $\tan 30 = \frac{1}{\sqrt{3}}$, each half-edge must be $\frac{1}{\sqrt{3}}\ units$ long. The whole perimeter (comprising 12 of these half-edges) is therefore $\frac{12}{\sqrt{3}}$ (or $4\sqrt{3}$, if simplified). This is approximately 6.928.

Since the circumference of the circle is greater than the smaller perimeter but less than the larger, we can bound the value of π using:

$$6 < 2\pi < 4\sqrt{3} \quad \Rightarrow \quad 3 < \pi < 2\sqrt{3} \quad (this\ means\ \pi\ must\ be\ between\ 3\ and\ around\ 3.464)$$

Taking a mean average of the perimeters gives $2\sqrt{3} + 3 \approx 6.464$. This yields an estimate for π since the circumference of the circle is between the two perimeter lengths: $2\pi \approx 6.464 \Rightarrow \pi \approx 3.23$.

Using a similar approach, with a 100-sided shape, we can narrow down the value of π to an accuracy of $3\ d.p.$
It takes a more than 1000-sided shape to boost the accuracy by a further 2 decimal places.