# Fibonacci $n^{th}$ term formula

1.

Prove that $\sqrt{6+2\sqrt{5}} = 1+\sqrt{5}$.

2.

Determine $\sqrt{6-2\sqrt{5}}$.

3.

The Fibonacci sequence is defined recursively (bit by bit) using the rule:

$$Fib(n) = Fib(n-1) + Fib(n-2) \quad where \quad Fib(1) = 1 \quad and \quad Fib(2) = 1$$

Using this formula, write the first 10 terms of the sequence.

4.

It can be shown that the $n^{th}$ term of this sequence is given by:

$$Fib(n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

Using proof by induction, and the results from questions 1 and 2, prove this formula.

Note: since the recursive formula depends on the previous *two* terms, it will be necessary to assume true for $n = k$ and $n = k+1$, then demonstrate true for the case $n = k+2$. Finally, show that the formula is true for $n = 1$ and $n = 2$.

# Fibonacci $n^{th}$ term formula

1.

Prove that $\sqrt{6+2\sqrt{5}} = 1+\sqrt{5}$.

$$(1+\sqrt{5})^2 = 1+2\sqrt{5}+5 = 6+2\sqrt{5} \quad \Rightarrow \quad \sqrt{6+2\sqrt{5}} = 1+\sqrt{5}$$

2.

Determine $\sqrt{6-2\sqrt{5}}$.

$$(1-\sqrt{5})^2 = 1-2\sqrt{5}+5 = 6-2\sqrt{5} \quad \Rightarrow \quad \sqrt{6-2\sqrt{5}} = 1-\sqrt{5}$$

3.

The Fibonacci sequence is defined recursively (bit by bit) using the rule:

$$Fib(n) = Fib(n-1) + Fib(n-2) \quad where \quad Fib(1) = 1 \quad and \quad Fib(2) = 1$$

Using this formula, write the first 10 terms of the sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

4.

It can be shown that the $n^{th}$ term of this sequence is given by:

$$Fib(n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

Using proof by induction, and the results from questions 1 and 2, prove this formula.

**Assume true for $n = k$ and $n = k+1$. Then: $Fib(k+2) = Fib(k+1) + Fib(k)$**

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(1+\frac{1-\sqrt{5}}{2}\right)\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{6-2\sqrt{5}}{4}\right)\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right\} \; for \; n = k+2.$$

**Therefore true for $n = k+2$.**

**When $n = 1$:**

$$\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right\} = \frac{1}{\sqrt{5}}\left\{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right\} = \frac{1}{\sqrt{5}}\{\sqrt{5}\} = 1 = Fib(1)$$

**When $n = 2$:**

$$\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}\right\} = \frac{1}{\sqrt{5}}\left\{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}\right\} = 1 = Fib(2)$$

**Therefore true for $n = 1$ and $n = 2$. By induction, true for all integers $n \geq 1$.**