Alternate C3 Exam *Based on the questions from AQA MPC3 June 2014*

1. Use Simpson's rule, with seven ordinates (six strips), to calculate an estimate for

$$\int_0^{\frac{\pi}{2}} x^2 \cos x \ dx$$

Give your answer to four significant figures.

[4 marks]

2. A curve has equation $y = \frac{\ln(e-2x)}{2}$

a) Find $\frac{dy}{dx}$.

[2 marks]

b) Find an equation of the normal to the curve $y = \frac{\ln(e-2x)}{2}$ at the point on the curve where $x = -e$.

[4 marks]

c) The curve $y = \frac{\ln(e-2x)}{2}$ crosses the line $y = x$ at $x = \alpha$.

i. Show that α lies between 0 and 1.

[2 marks]

ii. Use the recurrence relation $x_{n+1} = \frac{\ln(e-2x_n)}{2}$ with $x_1 = 0$ to find the values of x_2 and x_3 to 4 significant figures.

[2 marks]

3.

a) i. Differentiate $(3 - x^3)^{\frac{3}{2}}$ with respect to x.

[2 marks]

ii. Given that $y = e^{\frac{x}{2}}(3 - x^3)^{\frac{3}{2}}$, find the **exact value** of $\frac{dy}{dx}$ when $x = 0$.

[3 marks]

b) A curve has equation $y = \frac{x-1}{x^2+1}$. Use the quotient rule to find the **exact values** of the x coordinates of the stationary points of the curve.

[7 marks]

4.

a) Describe the graph transformation which maps $y = f(x-1)$ onto $y = f(x+3)$.

[2 marks]

b) Describe the graph transformation which maps $y = f(2x)$ onto $y = f(4x-8)$.

[4 marks]

c) Find the coordinates of the image of the point $P(3, -2)$ under the transformation described in b).

[2 marks]

5. The functions f and g are defined with their respective domains by

$$f(x) = x^2 + 5x - 6, \quad for\ x > -6$$
$$g(x) = |x + 10|, \quad for\ all\ real\ values\ of\ x$$

a) Find the range of f.

[2 marks]

b) The inverse of f is f^{-1}.

Find $f^{-1}(x)$. Give your answer in its simplest form.

[4 marks]

c) i. Find $fg(x)$.

[1 mark]

ii. Solve the equation $fg(x) = 8$

[6 marks]

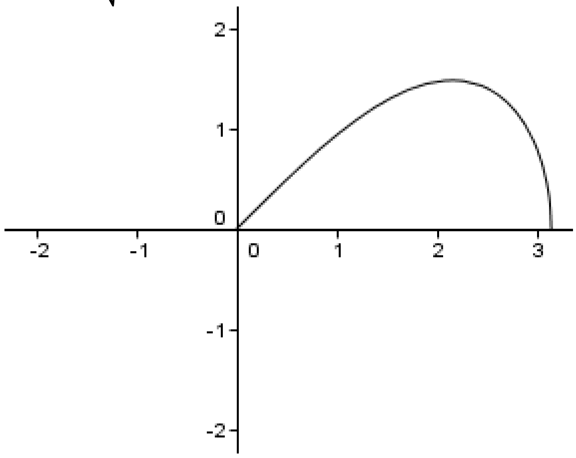
6.

a) By using integration by parts twice, find

$$\int x^2 \cos\frac{x}{2}\, dx$$

[6 marks]

b) The curve below has equation $y = x\sqrt{\cos\frac{x}{2}}$, for $0 \le x \le \pi$.



The region bounded by the curve and the x-axis is rotated through 2π radians about the x axis to generate a solid. Find the exact value of the volume of the solid generated.

[3 marks]

7. Use the substitution $u = 4 - x^4$ to find the exact value of

$$\int_0^1 \frac{x^7}{4 - x^4}\, dx$$

[6 marks]

8.

a) Show that the expression $\frac{1-\cos x}{\sin x} + \frac{\sin x}{1-\cos x}$ can be written as $2\,\mathrm{cosec}\, x$.

[4 marks]

b) Hence solve the equation

$$\frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} = 3\cot^2 x - 5$$

giving the values of x to the nearest degree in the interval $0° \le x < 360°$.

[6 marks]

c) Hence solve the equation

$$\frac{1 - \cos\left(\frac{\theta}{2} - 20°\right)}{\sin\left(\frac{\theta}{2} - 20°\right)} + \frac{\sin\left(\frac{\theta}{2} - 20°\right)}{1 - \cos\left(\frac{\theta}{2} - 20°\right)} = 3\cot^2\left(\frac{\theta}{2} - 20°\right) - 5$$

giving the values of θ to the nearest degree in the interval $0° \le \theta < 360°$.

[3 marks]

[Total: **75 marks**]

Alternate C3 Exam SOLUTIONS

1. *Use Simpson's rule, with seven ordinates (six strips), to calculate an estimate for*

$$\int_0^{\frac{\pi}{2}} x^2 \cos x \ dx$$

Give your answer to four significant figures.

[4 marks]

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$x_0 = 0 \qquad y_0 = 0^2 \cos 0 = 0$$

$$x_1 = \frac{\pi}{12} \qquad y_1 = \left(\frac{\pi}{12}\right)^2 \cos\frac{\pi}{12} = 0.06620 \ldots$$

$$x_2 = \frac{\pi}{6} \qquad y_2 = \left(\frac{\pi}{6}\right)^2 \cos\frac{\pi}{6} = 0.23742 \ldots$$

$$x_3 = \frac{\pi}{4} \qquad y_3 = \left(\frac{\pi}{4}\right)^2 \cos\frac{\pi}{4} = 0.43617 \ldots$$

$$x_4 = \frac{\pi}{3} \qquad y_4 = \left(\frac{\pi}{3}\right)^2 \cos\frac{\pi}{3} = 0.54831 \ldots$$

$$x_5 = \frac{5\pi}{12} \qquad y_5 = \left(\frac{5\pi}{12}\right)^2 \cos\frac{5\pi}{12} = 0.44347 \ldots$$

$$x_6 = \frac{\pi}{2} \qquad y_6 = \left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2} = 0$$

$$\int_0^{\frac{\pi}{2}} x^2 \cos x \ dx \approx \frac{1}{3} h\{(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\}$$

$$= \frac{1}{3}\left(\frac{\pi}{12}\right)\{(0+0) + 4(0.06620 \ldots + 0.43617 \ldots + 0.44347 \ldots) + 2(0.23742 \ldots + 0.54831 \ldots)\}$$

$$= 0.467305112 \ldots = \mathbf{0.4673 \ to \ 4 \ s.f.}$$

2. *A curve has equation* $y = \frac{ln(e-2x)}{2}$

a) Find $\frac{dy}{dx}$.

[2 marks]

b) Find an equation of the normal to the curve $y = \frac{ln(e-2x)}{2}$ *at the point on the curve where* $x = -e$.

[4 marks]

c) The curve $y = \frac{ln(e-2x)}{2}$ *crosses the line* $y = x$ *at* $x = \alpha$.

i. Show that α *lies between* 0 *and* 1.

[2 marks]

ii. Use the recurrence relation $x_{n+1} = \frac{ln(e-2x_n)}{2}$ *with* $x_1 = 0$ *to find the values of* x_2 *and* x_3 *to* 4 *significant figures.*

[2 marks]

2.

a)

Using chain rule:

$$y = \frac{\ln(e-2x)}{2} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{e-2x} \times -2 = \frac{-1}{e-2x} = \frac{\mathbf{1}}{\mathbf{2x - e}}$$

b)

$$\left(\frac{dy}{dx}\right)_{x=-e} = \frac{1}{2(-e)-e} = \frac{1}{-3e} \quad \Longrightarrow \quad Gradient\ of\ normal = 3e$$

$$y_{x=-e} = \frac{\ln\big(e-2(-e)\big)}{2} = \frac{\ln 3e}{2}$$

$$y - y_1 = m(x - x_1) \quad \Longrightarrow \quad normal: \ \ y - \frac{\ln 3e}{2} = 3e(x - -e)$$

Or:

$$\boldsymbol{y = 3ex + 3e^2 + \frac{1}{2}(1 + \ln 3)}$$

c)

i.

$y = x$ crosses $y = \frac{\ln(e-2x)}{2}$ when $x = \frac{\ln(e-2x)}{2} \quad \Longleftrightarrow \quad \frac{\ln(e-2x)}{2} - x = 0$

$$\frac{\ln(e-0)}{2} - 0 = \frac{1}{2} > 0 \quad and \quad \frac{\ln(e-2)}{2} - 1 \approx -1.1654 \ldots < 0$$

Since there is a change of sign from $x = 0$ to $x = 1$, **the root $\boldsymbol{\alpha}$ satisfies $\mathbf{0 < \boldsymbol{\alpha} < 1}$**.

ii.

$$x_1 = 0 \quad \Longrightarrow \quad x_2 = \frac{\ln(e-0)}{2} = \frac{1}{2} = \mathbf{0.5000\ to\ 4\ s.f.} \quad \Longrightarrow \quad x_3 = \frac{\ln\big(e - 2(0.5)\big)}{2} = \mathbf{0.2707\ to\ 4\ s.f.}$$

3.

a) i. Differentiate $(3-x^3)^{\frac{3}{2}}$ *with respect to* x.

[2 marks]

ii. Given that $y = e^{\frac{x}{2}}(3-x^3)^{\frac{3}{2}}$, *find the* ***exact value*** *of* $\frac{dy}{dx}$ *when* $x = 0$.

[3 marks]

b) A curve has equation $y = \frac{x-1}{x^2+1}$. *Use the quotient rule to find the* ***exact values*** *of the* x *coordinates of the stationary points of the curve.*

[7 marks]

3.

a)

i.

Using chain rule:

$$\frac{3}{2}(3-x^3)^{\frac{1}{2}}(-3x^2) = -\frac{9x^2(3-x^3)^{\frac{1}{2}}}{2}$$

ii.

Using product rule:

$$y = e^{\frac{x}{2}}(3-x^3)^{\frac{3}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = e^{\frac{x}{2}}\left(-\frac{9x^2(3-x^3)^{\frac{1}{2}}}{2}\right) + (3-x^3)^{\frac{3}{2}}\left(\frac{1}{2}e^{\frac{x}{2}}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=0} = e^0\left(-\frac{9(0)(3-0)^{\frac{1}{2}}}{2}\right) + (3-0)^{\frac{3}{2}}\left(\frac{1}{2}e^0\right) = \boldsymbol{\frac{3\sqrt{3}}{2}}$$

b)

$$y = \frac{x-1}{x^2+1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(x^2+1)(1)-(x-1)(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad (x^2+1)(1)-(x-1)(2x) = 0 \quad \Rightarrow \quad -x^2+2x+1 = 0$$

$$\Rightarrow \quad x^2-2x-1 = 0 \quad \Rightarrow \quad x = \frac{2 \pm \sqrt{4+4}}{2} = \boldsymbol{1 \pm \sqrt{2}}$$

4.

a) Describe the graph transformation which maps $y = f(x-1)$ onto $y = f(x+3)$.

[2 marks]

b) Describe the graph transformation which maps $y = f(2x)$ onto $y = f(4x-8)$.

[4 marks]

c) Find the coordinates of the image of the point $P(3,-2)$ under the transformation described in b).

[2 marks]

4.

a)

Translation by vector $\begin{bmatrix} \mathbf{-4} \\ \mathbf{0} \end{bmatrix}$.

b)

Stretch in the x direction by scale factor $\frac{1}{2}$ followed by a translation by vector $\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \end{bmatrix}$.

OR:

Translation by vector $\begin{bmatrix} \mathbf{4} \\ \mathbf{0} \end{bmatrix}$ followed by a stretch in the x direction by scale factor $\frac{1}{2}$.

c)

$$(3,-2) \to \left(\frac{3}{2},-2\right) \to \left(\mathbf{\frac{7}{2}},\mathbf{-2}\right)$$

OR:

$$(3,-2) \to (7,-2) \to \left(\mathbf{\frac{7}{2}},\mathbf{-2}\right)$$

5. *The functions f and g are defined with their respective domains by*

$$f(x) = x^2 + 5x - 6, \quad for\ x > 1$$
$$g(x) = |x + 10|, \quad for\ all\ real\ values\ of\ x$$

a) Find the range of f.

[2 marks]

b) The inverse of f is f^{-1}.

Find $f^{-1}(x)$. Give your answer in its simplest form.

[4 marks]

c) i. Find $fg(x)$.

[1 mark]

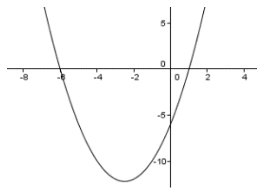
ii. Solve the equation $fg(x) = 8$

[6 marks]

5.

a)

$$f(x) = x^2 + 5x - 6 = (x + 6)(x - 1)$$



Therefore the graph $y = f(x)$ crosses the x-axis at the points $x = -6$ and $x = 1$ (and the y-axis at $y = -6$). From the graph it is apparent that the range is:

$$\boldsymbol{f(x) \geq 0}$$

b)

$$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 6 = \left(x + \frac{5}{2}\right)^2 - \frac{49}{4}$$

Let $y = f(x)$:

$$\left(x + \frac{5}{2}\right)^2 - \frac{49}{4} = y \quad \Rightarrow \quad \left(x + \frac{5}{2}\right)^2 = y + \frac{49}{4} \quad \Rightarrow \quad x + \frac{5}{2} = \pm\sqrt{y + \frac{49}{4}} \quad \Rightarrow \quad x = -\frac{5}{2} \pm \sqrt{y + \frac{49}{4}}$$

Reflecting in $y = x$:

$$y = -\frac{5}{2} \pm \sqrt{x + \frac{49}{4}}$$

However, since the range of f^{-1} is the domain of f, it is $f^{-1}(x) > 1$. Therefore we reject the $-ve$ root:

$$f^{-1}(x) = -\frac{5}{2} + \sqrt{x + \frac{49}{4}} \quad or \quad \boldsymbol{f^{-1}(x) = \frac{-5 + \sqrt{4x + 49}}{2}}$$

c)

i.

$$fg(x) = f(g(x)) = f(|x + 10|) = (|x + 10|)^2 + 5|x + 10| - 6 = \boldsymbol{(x + 10)^2 + 5|x + 10| - 6}$$

ii.

$$(x + 10)^2 + 5(x + 10) - 6 = 8 \quad or \quad (x + 10)^2 - 5(x + 10) - 6 = 8$$

$$(x + 10)^2 + 5(x + 10) - 14 = 0 \quad \Rightarrow \quad x^2 + 20x + 100 + 5x + 50 - 14 = 0 \quad \Rightarrow \quad x^2 + 25x + 136 = 0$$

$$\Rightarrow \quad (x + 8)(x + 17) = 0 \quad \Rightarrow \quad x = -8 \quad or \quad x = -17$$

$$(x + 10)^2 - 5(x + 10) - 14 = 0 \quad \Rightarrow \quad x^2 + 20x + 100 - 5x - 50 - 15 = 0 \quad \Rightarrow \quad x^2 + 15x + 36 = 0$$

$\Rightarrow \quad (x + 3)(x + 12) = 0 \quad \Rightarrow \quad x = -4 \quad or \quad x = -11$. All solutions: $\boldsymbol{x = -17, -12, -8\ or -3}$

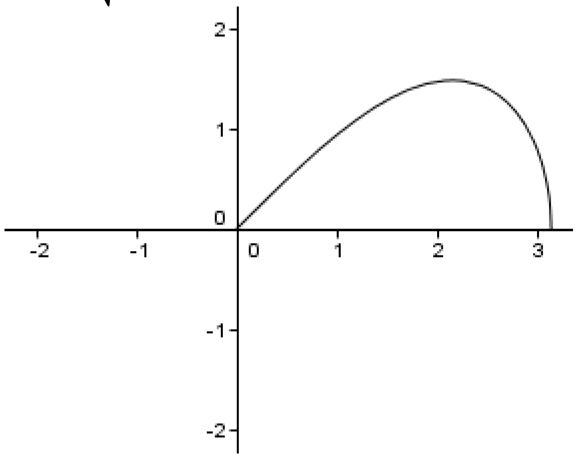
6.

a) By using integration by parts twice, find

$$\int x^2 \cos\frac{x}{2}\,dx$$

[6 marks]

b) The curve below has equation $y = x\sqrt{\cos\frac{x}{2}}$, for $0 \le x \le \pi$.



The region bounded by the curve and the x-axis is rotated through 2π radians about the x axis to generate a solid. Find the exact value of the volume of the solid generated.

[3 marks]

6.

a)

Integrating $\int x^2 \cos\frac{x}{2}\,dx$ by parts:

$$u = x^2 \qquad \frac{dv}{dx} = \cos\frac{x}{2}$$

$$\frac{du}{dx} = 2x \qquad v = 2\sin\frac{x}{2}$$

$$\int x^2 \cos\frac{x}{2}\,dx = 2x^2\sin\frac{x}{2} - \int 4x\sin\frac{x}{2}\,dx = 2x^2\sin\frac{x}{2} - 4\int x\sin\frac{x}{2}\,dx$$

Integrating $\int x\sin\frac{x}{2}\,dx$ by parts:

$$u = x \qquad \frac{dv}{dx} = \sin\frac{x}{2}$$

$$\frac{du}{dx} = 1 \qquad v = -2\cos\frac{x}{2}$$

$$\int x\sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int -2\cos\frac{x}{2}\,dx = -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + K$$

Substituting into expression for $\int x^2\cos\frac{x}{2}\,dx$ and incorporating the constant of integration:

$$\int x^2\cos\frac{x}{2}\,dx = 2x^2\sin\frac{x}{2} - 4\left(-2x\cos\frac{x}{2} + 4\sin\frac{x}{2}\right) + C = \boldsymbol{2x^2\sin\frac{x}{2} + 8x\cos\frac{x}{2} - 16\sin\frac{x}{2} + C}$$

b)

$$V = \pi\int_0^\pi y^2\,dx = \pi\int_0^\pi x^2\cos\frac{x}{2}\,dx = \pi\left[2x^2\sin\frac{x}{2} + 8x\cos\frac{x}{2} - 16\sin\frac{x}{2}\right]_0^\pi = \pi[(2\pi^2 - 16) - (0)]$$

$$\Rightarrow \quad \boldsymbol{V = 2\pi(\pi^2 - 8)}$$

7. *Use the substitution $u = 4 - x^4$ to find the exact value of*

$$\int_0^1 \frac{x^7}{4 - x^4}\, dx$$

[6 marks]

7.

$$u = 4 - x^4 \quad \Longrightarrow \quad \frac{du}{dx} = -4x^3 \quad \Longrightarrow \quad -\frac{1}{4}\, du = x^3\, dx$$

$$x = 0 \quad \Longrightarrow \quad u = 4 - 0 = 4 \quad and \quad x = 1 \quad \Longrightarrow \quad u = 4 - 1^4 = 3$$

$$\int_0^1 \frac{x^7}{4 - x^4}\, dx = -\frac{1}{4}\int_4^3 \frac{x^4}{u}\, du = -\frac{1}{4}\int_4^3 \frac{4 - u}{u}\, du = -\frac{1}{4}\int_4^3 \frac{4}{u} - 1\, du = -\frac{1}{4}[4\ln u - u]_4^3$$

$$= -\frac{1}{4}[(4\ln 3 - 3) - (4\ln 4 - 4)] = -\frac{1}{4}[4(\ln 3 - \ln 4) + 1] = \mathbf{\ln 4 - \ln 3 - \frac{1}{4}}$$

8.

a) Show that the expression $\frac{1-\cos x}{\sin x}+\frac{\sin x}{1-\cos x}$ *can be written as* $2\,cosec\,x$.

[4 marks]

b) Hence solve the equation

$$\frac{1-\cos x}{\sin x}+\frac{\sin x}{1-\cos x}=3\cot^2 x-5$$

giving the values of x *to the nearest degree in the interval* $0° \le x < 360°$.

[6 marks]

c) Hence solve the equation

$$\frac{1-\cos\left(\frac{\theta}{2}-20°\right)}{\sin\left(\frac{\theta}{2}-20°\right)}+\frac{\sin\left(\frac{\theta}{2}-20°\right)}{1-\cos\left(\frac{\theta}{2}-20°\right)}=3\cot^2\left(\frac{\theta}{2}-20°\right)-5$$

giving the values of θ *to the nearest degree in the interval* $0° \le \theta < 360°$.

[3 marks]

*[Total: **75 marks**]*

8.

a)

$$\frac{1-\cos x}{\sin x}+\frac{\sin x}{1-\cos x}=\frac{(1-\cos x)^2}{\sin x\,(1-\cos x)}+\frac{\sin^2 x}{\sin x\,(1-\cos x)}$$

$$=\frac{(1-\cos x)^2+\sin^2 x}{\sin x\,(1-\cos x)}=\frac{1-2\cos x+\cos^2 x+\sin^2 x}{\sin x\,(1-\cos x)}=\frac{2-2\cos x}{\sin x\,(1-\cos x)}=\frac{2}{\sin x}=\mathbf{2\,cosec\,x}$$

b)

$$2\operatorname{cosec} x=3\cot^2 x-5 \quad\Longrightarrow\quad 2\operatorname{cosec} x=3(\operatorname{cosec}^2 x-1)-5 \quad\Longrightarrow\quad 3\operatorname{cosec}^2 x-2\operatorname{cosec} x-8=0$$

$$\Longrightarrow\quad (3\operatorname{cosec} x+4)(\operatorname{cosec} x-2)=0 \quad\Longrightarrow\quad \operatorname{cosec} x=-\frac{4}{3}\quad or\quad \operatorname{cosec} x=2$$

Solving in the interval $0° \le x < 360°$:

$$\sin x=-\frac{3}{4}\quad or\quad \sin x=\frac{1}{2}\quad\Longrightarrow\quad x=228.59\ldots°, 311.40\ldots°\quad or\quad x=30°, 150°$$

All solutions: $\boldsymbol{x}=\mathbf{30}°, \mathbf{150}°, \mathbf{229}°, \mathbf{311}°$ to the nearest degree.

c)

$$0° \le \theta < 360 \quad\Longrightarrow\quad -20° \le \frac{\theta}{2}-20° < 160°$$

$$\frac{\theta}{2}-20=30°\ \ or\ \ 150° \quad\Longrightarrow\quad \frac{\theta}{2}=50\ \ or\ \ 170 \quad\Longrightarrow\quad \boldsymbol{\theta}=\mathbf{100}°\ \ \boldsymbol{or}\ \ \mathbf{340}°$$