AQA MPC3 Core 3 Mathematics 10 June 2014

Question Paper and Worked Solutions

Please note, this document represents my own solutions to the questions, is entirely unofficial and is not related to the mark scheme (which I have not seen). Therefore, while it should help you see how to do the questions, it won't include every valid method or give you a break down of the mark allocation. If you spot any errors, or think you have found a better solution, please [email me] so I can update it.

1 Use Simpson's rule, with five ordinates (four strips), to calculate an estimate for

$$\int_0^{\pi} x^{\frac{1}{2}} \sin x \; dx$$

Give your answer to four significant figures.

[4 marks]

1.
Use Simpson's rule, with seven ordinates (six strips), to calculate an estimate for

$$\int_0^{\frac{3\pi}{2}} x^2 \cos x \; dx$$

Give your answer to four significant figures.

[4 marks]

2 A curve has equation $y = 2\ln(2e - x)$.

(a) Find $\frac{dy}{dx}$.

[2 marks]

(b) Find an equation of the normal to the curve $y = 2\ln(2e - x)$ at the point on the curve where $x = e$.

[4 marks]

(c) The curve $y = 2\ln(2e - x)$ intersects the line $y = x$ at a single point, where $x = \alpha$.

(i) Show that α lies between 1 and 3.

[2 marks]

(ii) Use the recurrence relation

$$x_{n+1} = 2\ln(2e - x_n)$$

with $x_1 = 1$ to find the values of x_2 and x_3, giving your answers to three decimal places.

[2 marks]

(iii) **Figure 1**, on the opposite page, shows a sketch of parts of the graphs of $y = 2\ln(2e - x)$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis.

[2 marks]

2.

A curve has equation $y = 5\ln(e + 2x)$

a)

Find $\frac{dy}{dx}$.

[2 marks]

b)

Find an equation of the normal to the curve $y = 5\ln(e + 2x)$ at the point on the curve where $x = 2e$.

[4 marks]

c)

The curve $y = 5\ln(e + 2x)$ crosses the line $y = x$ at $x = \alpha$.

i.

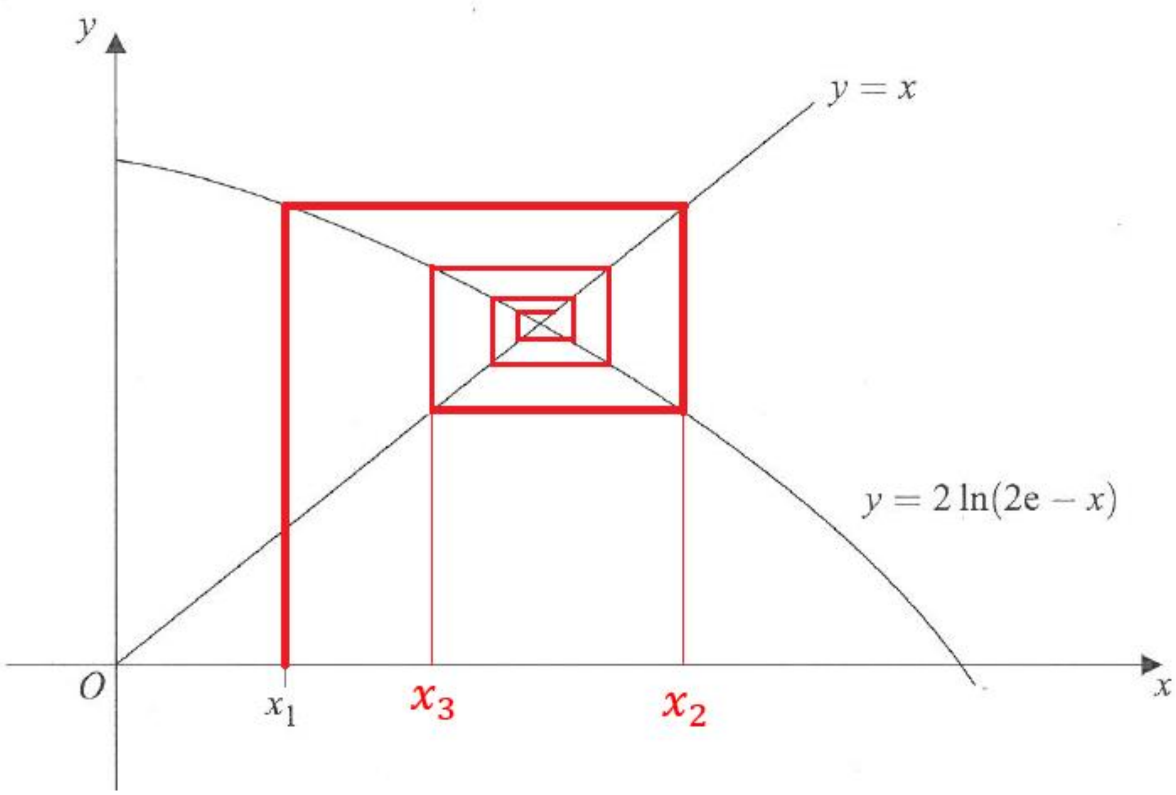
Show that α lies between 0 and -2.

[2 marks]

ii.

iii.

(c)(iii)



3 (a) (i) Differentiate $(x^2+1)^{\frac{5}{2}}$ with respect to x.

[2 marks]

(ii) Given that $y = e^{2x}(x^2+1)^{\frac{5}{2}}$, find the value of $\frac{dy}{dx}$ when $x = 0$.

[3 marks]

(b) A curve has equation $y = \frac{4x-3}{x^2+1}$. Use the quotient rule to find the x-coordinates of the stationary points of the curve.

[5 marks]

3.

a)

i.

Using chain rule: $f(x) = (x^2+1)^{\frac{5}{2}} \implies f'(x) = \frac{5}{2}(x^2+1)^{\frac{3}{2}}(2x)$

ii.

Using product rule: $y = e^{2x}(x^2+1)^{\frac{5}{2}} \implies \frac{dy}{dx} = e^{2x}\frac{5}{2}(x^2+1)^{\frac{3}{2}}(2x) + (x^2+1)^{\frac{5}{2}}(2e^{2x})$

$$x = 0 \implies \frac{dy}{dx} = e^0\frac{5}{2}(1)^{\frac{3}{2}}(0) + (1)^{\frac{5}{2}}(2e^0) = 0 + 2 = \mathbf{2}$$

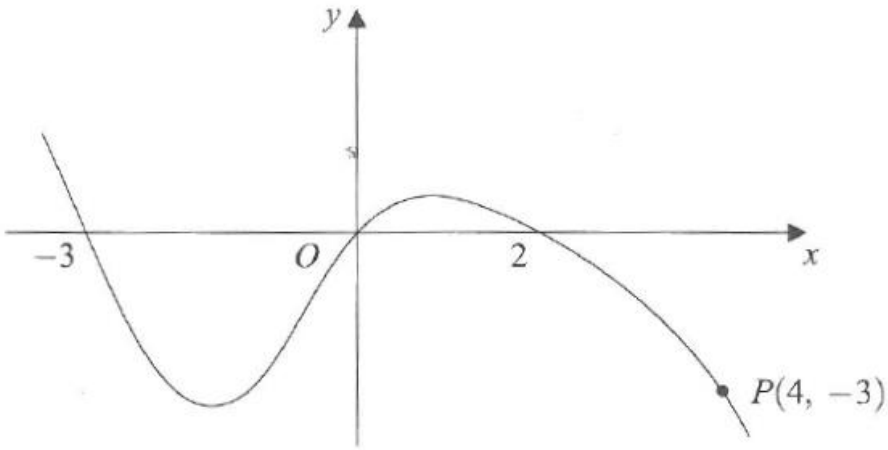
b)

$$y = \frac{4x-3}{x^2+1} \implies \frac{dy}{dx} = \frac{(x^2+1)(4)-(4x-3)(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = 0 \implies \frac{(x^2+1)(4)-(4x-3)(2x)}{(x^2+1)^2} = 0 \implies (x^2+1)(4)-(4x-3)(2x) = 0$$

$$\implies 4x^2+4-8x^2+6x = 0 \implies 2x^2-3x-2 = 0 \implies (2x+1)(x-2) = 0 \implies \boldsymbol{x = -\frac{1}{2}} \quad \boldsymbol{or} \quad \boldsymbol{x = 2}$$

4 The sketch shows part of the curve with equation $y = f(x)$.



(a) On **Figure 2** below, sketch the curve with equation $y = -|f(x)|$.

[3 marks]

(b) On **Figure 3** on the page opposite, sketch the curve with equation $y = f(|2x|)$.

[2 marks]

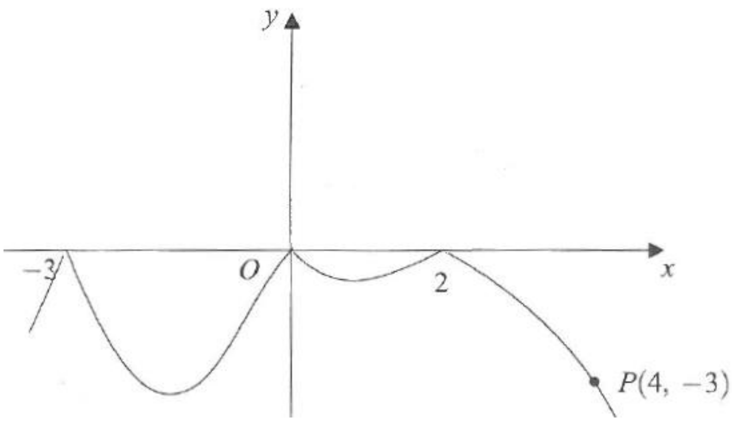
(c) (i) Describe a sequence of two geometrical transformations that maps the graph of $y = f(x)$ onto the graph of $y = f(2x + 2)$.

[4 marks]

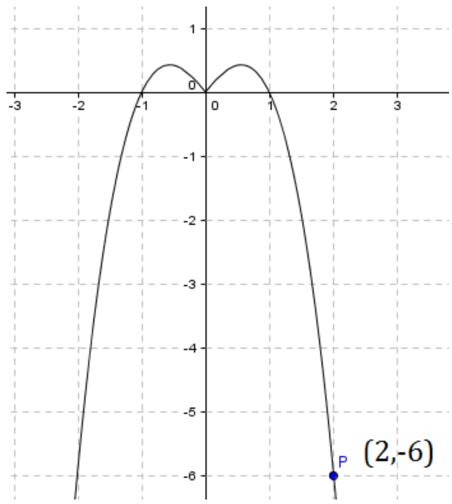
(ii) Find the coordinates of the image of the point $P(4, -3)$ under the sequence of transformations given in part **(c)(i)**.

[2 marks]

a)



b)



c)

i.

$$f(x) \to f(x+2) \to f(2x+2)$$

$$Translation\ of\ \begin{bmatrix}-2\\0\end{bmatrix}\ followed\ by\ a\ stretch\ of\ scale\ factor \frac{1}{2}\ in\ the\ x\ direction$$

OR:

$$f(x) \to f(2x) \to f\big(2(x+1)\big) = f(2x+2)$$

$$Stretch\ of\ scale\ factor \frac{1}{2}\ in\ the\ x\ direction\ followed\ by\ a\ translation\ of\ \begin{bmatrix}-1\\0\end{bmatrix}$$

ii.

$$(4,-3) \to (2,-3) \to (\mathbf{1}, \mathbf{-3})$$

5 The functions f and g are defined with their respective domains by

$$f(x) = x^2 - 6x + 5, \quad \text{for } x \geqslant 3$$

$$g(x) = |x - 6|, \quad \text{for all real values of } x$$

(a) Find the range of f.

[2 marks]

(b) The inverse of f is f^{-1}.

Find $f^{-1}(x)$. Give your answer in its simplest form.

[4 marks]

(c) (i) Find $gf(x)$.

[1 mark]

(ii) Solve the equation $gf(x) = 6$.

[4 marks]

5.

a)

$$f(x) = x^2 - 6x + 5 = (x-3)^2 - 9 + 5 = (x-3)^2 - 4$$

$$Domain: \quad x \geq 3 \quad \Longrightarrow \quad Range: \quad f(x) \geq -4$$

b)

$$y = f(x) \quad \Longrightarrow \quad x = f^{-1}(y)$$

$$y = (x-3)^2 - 4 \quad \Longrightarrow \quad \sqrt{y+4} = x - 3 \quad \Longrightarrow \quad x = 3 + \sqrt{y+4} \quad \Longrightarrow \quad \boldsymbol{f^{-1}(x) = 3 + \sqrt{x+4}}$$

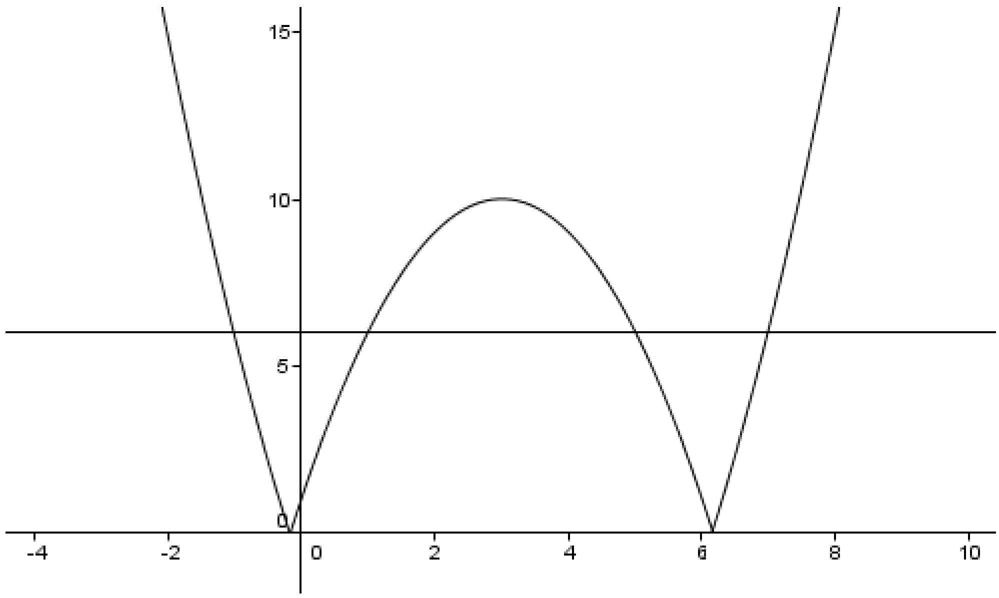
Note: only the positive square root is required since $x \geq 3$.

c)

i.

$$gf(x) = g\big(f(x)\big) = g((x-3)^2 - 4) = |(x-3)^2 - 4 - 6| = \boldsymbol{|(x-3)^2 - 10|} \quad \boldsymbol{or} \quad \boldsymbol{gf(x) = |x^2 - 6x - 1|}$$

ii.



$$|(x-3)^2 - 10| = 6 \quad \Longrightarrow \quad (x-3)^2 - 10 = 6 \quad or \quad (x-3)^2 - 10 = -6$$

$$(x-3)^2 - 10 = 6 \quad \Longrightarrow \quad x = 3 \pm 4 \quad \Longrightarrow \quad x = 7 \quad or \quad x = -1$$

$$(x-3)^2 - 10 = -6 \quad \Longrightarrow \quad x = 3 \pm 2 \quad \Longrightarrow \quad x = 5 \quad or \quad x = 1$$

However, the domain for f is $x \geq 3$, therefore the only valid solutions are: $\boldsymbol{x = 5 \quad and \quad x = 7}$.

6 (a) By using integration by parts twice, find

$$\int x^2 \sin 2x \, dx$$

[6 marks]

(b) A curve has equation $y = x\sqrt{\sin 2x}$, for $0 \leqslant x \leqslant \frac{\pi}{2}$.

The region bounded by the curve and the x-axis is rotated through 2π radians about the x-axis to generate a solid.

Find the exact value of the volume of the solid generated.

[3 marks]

6.

a)

$$\int x^2 \sin 2x \ dx \qquad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$u = x^2 \quad \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = \sin 2x \quad v = -\frac{1}{2}\cos 2x$$

$$\int x^2 \sin 2x \ dx = -\frac{x^2}{2}\cos 2x - \int -x\cos 2x \ dx = -\frac{x^2}{2}\cos 2x + \int x\cos 2x \ dx + A$$

Now, integrating $\int x\cos 2x \ dx$ by parts:

$$\int x\cos 2x \ dx \qquad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$u = x \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos 2x \quad v = \frac{1}{2}\sin 2x$$

$$\int x\cos 2x \ dx = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x \ dx = \frac{x}{2}\sin 2x - -\frac{1}{4}\cos 2x = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + B$$

Finally, substituting back into the original:

$$\int x^2 \sin 2x \ dx = -\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x = \boldsymbol{\frac{2x\sin 2x + (1-2x^2)\cos 2x}{4} + C}$$

b)

$$V = \pi\int_0^{\frac{\pi}{2}}\left(x\sqrt{\sin 2x}\right)^2 dx = \pi\int_0^{\frac{\pi}{2}} x^2\sin 2x \ dx = \pi\left[\frac{2x\sin 2x + (1-2x^2)\cos 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}\left[\left(\frac{\pi^2}{2} - 1\right) - (1)\right] = \boldsymbol{\frac{\pi}{8}(\pi^2 - 4)}$$

7 Use the substitution $u = 3 - x^3$ to find the exact value of $\int_0^1 \frac{x^5}{3 - x^3}\,dx$.

[6 marks]

7.

$$u = 3 - x^3 \implies du = -3x^2\,dx$$

$$\int_0^1 \frac{x^5}{3-x^3}\,dx = -\frac{1}{3}\int_3^2 \frac{3-u}{u}\,du = -\frac{1}{3}\int_3^2 \frac{3}{u} - 1\,du = -\frac{1}{3}[3\ln u - u]_3^2 = -\frac{1}{3}[(3\ln 2 - 2) - (3\ln 3 - 3)]$$

$$= -\frac{1}{3}\left[3\ln\frac{2}{3} + 1\right] = -\ln\frac{2}{3} - \frac{1}{3} = \mathbf{\ln\frac{3}{2} - \frac{1}{3}}$$

8 (a) Show that the expression $\frac{1 - \sin x}{\cos x} + \frac{\cos x}{1 - \sin x}$ can be written as $2\sec x$.

[4 marks]

(b) Hence solve the equation

$$\frac{1 - \sin x}{\cos x} + \frac{\cos x}{1 - \sin x} = \tan^2 x - 2$$

giving the values of x to the nearest degree in the interval $0° \leqslant x < 360°$.

[6 marks]

(c) Hence solve the equation

$$\frac{1 - \sin(2\theta - 30°)}{\cos(2\theta - 30°)} + \frac{\cos(2\theta - 30°)}{1 - \sin(2\theta - 30°)} = \tan^2(2\theta - 30°) - 2$$

giving the values of θ to the nearest degree in the interval $0° \leqslant \theta \leqslant 180°$.

[2 marks]

8.

a)

$$\frac{1-\sin x}{\cos x} + \frac{\cos x}{1-\sin x} = \frac{(1-\sin x)^2 + \cos^2 x}{\cos x\,(1-\sin x)} = \frac{1 - 2\sin x + \sin^2 x + \cos^2 x}{\cos x\,(1-\sin x)} = \frac{2(1-\sin x)}{\cos x\,(1-\sin x)} = \mathbf{2\sec x}$$

b)

$$\frac{1-\sin x}{\cos x} + \frac{\cos x}{1-\sin x} = \tan^2 x - 2 \implies 2\sec x = \tan^2 x - 2 \implies 2\sec x = (\sec^2 x - 1) - 2$$

$$\implies 2\sec x = \sec^2 x - 3 \implies \sec^2 x - 2\sec x - 3 = 0 \implies (\sec x + 1)(\sec x - 3) = 0$$

$$\implies \sec x = -1 \quad or \quad \sec x = 3 \implies \cos x = -1 \quad or \quad \cos x = \frac{1}{3}$$

$$\cos x = -1 \implies x = 180° \quad and \quad \cos x = \frac{1}{3} \implies x = \cos^{-1}\frac{1}{3} \approx 71° \quad or \quad 289°$$

$$\mathbf{x = 71° \quad or \quad 180° \quad or \quad 289°}$$

c)

$$\frac{1-\sin(2\theta - 30)}{\cos(2\theta - 30)} + \frac{\cos(2\theta - 30)}{1 - \sin(2\theta - 30)} = \tan^2(2\theta - 30) - 2 \implies 2\theta - 30 = x$$

$$\implies \theta = \frac{180 + 30}{2} = 105° \quad or \quad \theta = \frac{70.52\ldots + 30}{2} \approx 50° \quad or \quad \theta = \frac{289.47\ldots + 30}{2} \approx 160°$$

$$\boldsymbol{\theta = 50° \quad or \quad 105° \quad or \quad 160°}$$