AQA MFP2 Further Pure 2 Further Mathematics 18 June 2014

Question Paper and Worked Solutions

Please note, this document represents my own solutions to the questions, is entirely unofficial and is not related to the mark scheme (which I have not seen). Therefore, while it should help you see how to do the questions, it won't include every valid method or give you a break down of the mark allocation. If you spot any errors, or think you have found a better solution, please [email me] so I can update it.

1 (a) Express $-9\mathrm{i}$ in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

[2 marks]

(b) Solve the equation $z^4 + 9\mathrm{i} = 0$, giving your answers in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

[5 marks]

1.

a)

$$|-9i| = 9 \quad \arg(-9i) = -\frac{\pi}{2} \quad \Rightarrow \quad -9i = \boldsymbol{9e^{-\frac{\pi}{2}i}}$$

b)

$$z^4 + 9i = 0 \quad \Rightarrow \quad z = \sqrt[4]{-9i} = \left(9e^{\left(2n\pi - \frac{\pi}{2}\right)i}\right)^{\frac{1}{4}} = \sqrt{3}e^{\left(\frac{2n\pi}{4} - \frac{\pi}{8}\right)i} = \sqrt{3}e^{-\frac{\pi}{8}i}, \sqrt{3}e^{\left(\frac{2\pi}{4} - \frac{\pi}{8}\right)i}, \sqrt{3}e^{\left(\frac{4\pi}{4} - \frac{\pi}{8}\right)i}, \sqrt{3}e^{\left(\frac{6\pi}{4} - \frac{\pi}{8}\right)i}$$

$$= \sqrt{3}e^{-\frac{\pi}{8}i}, \sqrt{3}e^{-\frac{3\pi}{8}i}, \sqrt{3}e^{\frac{7\pi}{8}i}, \sqrt{3}e^{\frac{11\pi}{8}i} = \boldsymbol{\sqrt{3}e^{-\frac{\pi}{8}i}, \sqrt{3}e^{-\frac{3\pi}{8}i}, \sqrt{3}e^{\frac{7\pi}{8}i}, \sqrt{3}e^{\frac{-5\pi}{8}i}}$$

2 (a) Sketch, on the Argand diagram below, the locus L of points satisfying

$$\arg(z - 2\mathrm{i}) = \frac{2\pi}{3}$$

[3 marks]

(b) (i) A circle C, of radius 3, has its centre lying on L and touches the line $\mathrm{Im}(z) = 2$. Sketch C on the Argand diagram used in part **(a)**.

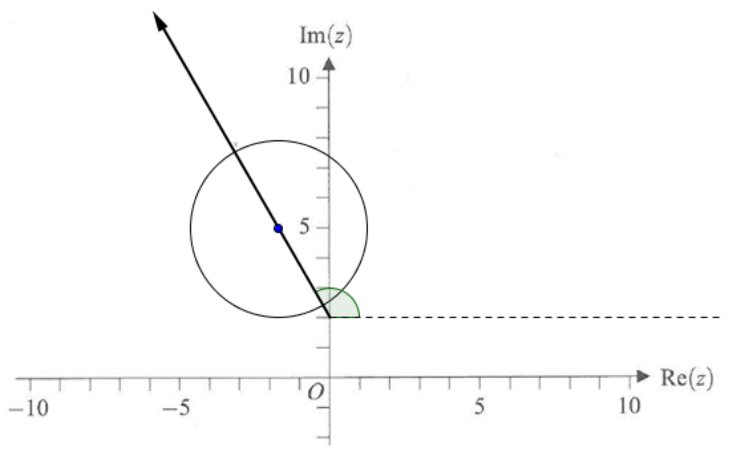
[2 marks]

(ii) Find the centre of C, giving your answer in the form $a + b\mathrm{i}$.

[3 marks]

2.

a) and b)i.



ii.

Forming a right-angled triangle with the lowest point of the circle, the centre of the circle and the point $-2i$:

$$\tan\frac{\pi}{3} = \frac{3}{x} \quad \Rightarrow \quad \sqrt{3} = \frac{3}{x} \quad \Rightarrow \quad x = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \Rightarrow \quad \boldsymbol{Coordinates\ of\ centre}: \ \boldsymbol{-\sqrt{3} + 5i}$$

3 (a) Express $(k+1)^2 + 5(k+1) + 8$ in the form $k^2 + ak + b$, where a and b are constants.

[1 mark]

(b) Prove by induction that, for all integers $n \geqslant 1$,

$$\sum_{r=1}^{n} r(r+1)\left(\frac{1}{2}\right)^{r-1} = 16 - \left(n^2 + 5n + 8\right)\left(\frac{1}{2}\right)^{n-1}$$

[6 marks]

3.

a)

$$(k+1)^2 + 5(k+1) + 8 = k^2 + 2k + 1 + 5k + 5 + 8 = \boldsymbol{k^2 + 7k + 14}$$

b)

Assume true for $\boldsymbol{n = k}$:

$$\sum_{r=1}^{k} r(r+1)\left(\frac{1}{2}\right)^{r-1} = 16 - (k^2 + 5k + 8)\left(\frac{1}{2}\right)^{k-1}$$

When $n = k + 1$:

$$\sum_{r=1}^{k+1} r(r+1)\left(\frac{1}{2}\right)^{r-1} = \sum_{r=1}^{k} r(r+1)\left(\frac{1}{2}\right)^{r-1} + (k+1)\big((k+1)+1\big)\left(\frac{1}{2}\right)^{(k+1)-1}$$

$$= 16 - (k^2 + 5k + 8)\left(\frac{1}{2}\right)^{k-1} + (k+1)(k+2)\left(\frac{1}{2}\right)^{k}$$

$$= 16 - \{2(k^2 + 5k + 8) - (k+1)(k+2)\}\left(\frac{1}{2}\right)^{k}$$

$$= 16 - \{2k^2 + 10k + 16 - (k^2 + 3k + 2)\}\left(\frac{1}{2}\right)^{(k+1)-1}$$

$$= 16 - \{k^2 + 7k + 14\}\left(\frac{1}{2}\right)^{(k+1)-1} = 16 - \{(k+1)^2 + 5(k+1) + 8\}\left(\frac{1}{2}\right)^{(k+1)-1} \quad \Rightarrow \quad \boldsymbol{true\ for\ n = k + 1}$$

When $n = 1$:

$$\sum_{r=1}^{1} r(r+1)\left(\frac{1}{2}\right)^{r-1} = (1)(2)\left(\frac{1}{2}\right)^{0} = 2 \quad and \quad 16 - (1^2 + 5(1) + 8)\left(\frac{1}{2}\right)^{0} = 2 \quad \Rightarrow \quad \boldsymbol{true\ for\ n = 1}$$

$$\boldsymbol{True\ for\ n = k \ \Rightarrow\ True\ for\ n = k + 1\ \ and\ \ True\ for\ n = 1 \ \Rightarrow\ By\ induction\ true\ for\ n \geq 1}$$

4 The roots of the equation

$$z^3 + 2z^2 + 3z - 4 = 0$$

are α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta + \beta\gamma + \gamma\alpha$.

[2 marks]

(ii) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = -2$.

[2 marks]

(b) Find the value of:

(i) $(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta)$;

[3 marks]

(ii) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$.

[4 marks]

(c) Find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$ and $\gamma + \alpha$.

[3 marks]

4.

a)

i.

$$\Sigma\alpha = -\frac{b}{a} = \mathbf{-2} \quad and \quad \Sigma\alpha\beta = \frac{c}{a} = \mathbf{3}$$

ii.

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta) = (-2)^2 - 2(3) = 4 - 6 = \mathbf{-2}$$

b)

i.

$$(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) = \beta^2 + \Sigma\alpha\beta + \gamma^2 + \Sigma\alpha\beta + \alpha^2 + \Sigma\alpha\beta = \Sigma\alpha^2 + 3\Sigma\alpha\beta = \mathbf{7}$$

ii.

$$(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = \alpha\beta\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2 + \alpha\beta\gamma = (\Sigma\alpha)(\Sigma\alpha\beta) - \alpha\beta\gamma$$

$$\alpha\beta\gamma = -\frac{d}{a} = 4 \quad and \quad \Sigma\alpha = -2 \quad and \quad \Sigma\alpha\beta = 3 \quad \Rightarrow \quad (\Sigma\alpha)(\Sigma\alpha\beta) - \alpha\beta\gamma = (-2)(3) - 4 = \mathbf{-10}$$

c)

$$\Sigma\alpha' = 2\Sigma\alpha = -4 = -\frac{b}{a} \qquad \Sigma\alpha'\beta' = 7 = \frac{c}{a} \qquad \alpha'\beta'\gamma' = -10 = -\frac{d}{a}$$

$$a = 1 \quad \Rightarrow \quad b = 4 \quad c = 7 \quad d = 10 \quad \Rightarrow \quad \boldsymbol{x^3 + 4x^2 + 7x + 10 = 0}$$

5 (a) Using the definition $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$, prove the identity

$$4\sinh^3\theta + 3\sinh\theta = \sinh 3\theta$$

[3 marks]

(b) Given that $x = \sinh\theta$ and $16x^3 + 12x - 3 = 0$, find the value of θ in terms of a natural logarithm.

[4 marks]

(c) Hence find the real root of the equation $16x^3 + 12x - 3 = 0$, giving your answer in the form $2^p - 2^q$, where p and q are rational numbers.

[2 marks]

5.

a)

$$4\sinh^3\theta + 3\sinh\theta = 4\left(\frac{1}{2}(e^{\theta} - e^{-\theta})\right)^3 + 3\left(\frac{1}{2}(e^{\theta} - e^{-\theta})\right) = \frac{1}{2}(e^{\theta} - e^{-\theta})^3 + \frac{3}{2}(e^{\theta} - e^{-\theta})$$

$$= \frac{1}{2}\{(e^{3\theta} - 3e^{\theta} + 3e^{-\theta} - e^{-\theta}) + (3e^{\theta} - 3e^{-\theta})\} = \frac{1}{2}(e^{3\theta} - e^{-3\theta}) = \mathbf{\sinh 3\boldsymbol{\theta}}$$

b)

$$16\sinh^3\theta + 12\sinh\theta - 3 = 0 \implies 4(\sinh 3\theta) - 3 = 0 \implies \sinh 3\theta = \frac{3}{4} \implies \frac{1}{2}(e^{3\theta} - e^{-3\theta}) = \frac{3}{4}$$

$$\implies e^{3\theta} - e^{-3\theta} = \frac{3}{2} \implies e^{6\theta} - 1 = \frac{3}{2}e^{3\theta} \implies 2e^{6\theta} - 3e^{3\theta} - 2 = 0 \implies (2e^{3\theta} + 1)(e^{3\theta} - 2) = 0$$

$$\implies e^{3\theta} = -\frac{1}{2} \text{ or } 2 \implies e^{3\theta} = 2 \implies 3\theta = \ln 2 \implies \boldsymbol{\theta = \frac{1}{3}\ln 2}$$

c)

$$\theta = \frac{1}{3}\ln 2 \implies x = \sinh\left(\frac{1}{3}\ln 2\right) = \frac{1}{2}\left(e^{\ln 2^{\frac{1}{3}}} - e^{\ln 2^{-\frac{1}{3}}}\right) = \frac{1}{2}\left(2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right) = \mathbf{2^{-\frac{1}{3}} - 2^{-\frac{4}{3}}}$$

6 (a) (i) Use De Moivre's Theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

[3 marks]

(ii) Write down a similar expression for $z^n + \frac{1}{z^n}$.

[1 mark]

(b) (i) Expand $\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2$ in terms of z.

[1 mark]

(ii) Hence show that

$$8\sin^2\theta\cos^2\theta = A + B\cos 4\theta$$

where A and B are integers.

[2 marks]

(c) Hence, by means of the substitution $x = 2\sin\theta$, find the exact value of

$$\int_1^2 x^2\sqrt{4 - x^2}\,\mathrm{d}x$$

[5 marks]

6.

a)

i.

$$z = \cos\theta + i\sin\theta \quad \Rightarrow \quad z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$and \quad z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos -n\theta + i\sin -n\theta = \cos n\theta - i\sin n\theta$$

$$\Rightarrow \quad z^n - \frac{1}{z^n} = z^n - z^{-n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = \boldsymbol{2i\sin n\theta}$$

ii.

$$z^n + \frac{1}{z^n} = z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = \mathbf{2\cos}\, \boldsymbol{n\theta}$$

b)

i.

$$\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2 = \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^2 + 2 + \frac{1}{z^2}\right) = z^4 + 2z^2 + 1 - 2z^2 - 4 - \frac{2}{z^2} + 1 + \frac{2}{z^2} + \frac{1}{z^4} = \boldsymbol{z^4 + \frac{1}{z^4} - 2}$$

ii.

$$8\sin^2\theta\cos^2\theta = 8\left(\frac{\left(z - \frac{1}{z}\right)^2}{(2i)^2}\right)\left(\frac{\left(z + \frac{1}{z}\right)^2}{2^2}\right) = -\frac{1}{2}\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2 = -\frac{1}{2}\left(z^4 + \frac{1}{z^4} - 2\right) = -\frac{1}{2}\left(z^4 + \frac{1}{z^4}\right) + 1$$

$$= -\frac{1}{2}(2\cos 4\theta) + 1 = \mathbf{1 - \cos 4}\boldsymbol{\theta}$$

c)

$$x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \qquad \int_1^2 x^2\sqrt{4 - x^2}\,dx = \int_{\sin^{-1}\frac{1}{2}}^{\sin^{-1}1} 4\sin^2\theta\sqrt{4 - 4\sin^2\theta}\,(2)\cos\theta\ d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\sin^2\theta\cos^2\theta\ d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 - \cos 4\theta\ d\theta = 2\left[\theta - \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\left(\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right) = \boldsymbol{\frac{\sqrt{3}}{4} + \frac{2\pi}{3}}$$

7 (a) Given that $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $x \neq 1$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$.

[4 marks]

(b) Hence, given that $x < 1$, show that $\tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4}$.

[3 marks]

7.

a)

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right) \implies \tan y = \frac{1+x}{1-x} \implies \sec^2 y \frac{dy}{dx} = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$\sec^2 y = \tan^2 y + 1 = \frac{(1+x)^2}{(1-x)^2} + 1 = \frac{(1+x)^2+(1-x)^2}{(1-x)^2} = \frac{2(1+x^2)}{(1-x)^2} \implies \sec^2 y \frac{dy}{dx} = \frac{2(1+x^2)}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{2}{(1-x)^2} \times \frac{(1-x)^2}{2(1+x^2)} = \frac{(1-x)^2}{(1-x)^2(1+x^2)} = \mathbf{\frac{1}{1+x^2}}$$

b)

$$\frac{dy}{dx} = \frac{1}{1+x^2} \implies \int 1\, dy = \int \frac{1}{1+x^2}\, dx \implies y = \tan^{-1} x + C$$

$$\implies \tan^{-1} x + C = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

$$x = 0 \implies \tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}(1) = \frac{\pi}{4} \quad \text{and} \quad x = 0 \implies \tan^{-1} x + C = \tan^{-1} 0 + C = C$$

$$\implies C = \frac{\pi}{4} \implies \tan^{-1} x + \frac{\pi}{4} = \tan^{-1}\left(\frac{1+x}{1-x}\right) \implies \mathbf{\tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4}}$$

8 A curve has equation $y = 2\sqrt{x-1}$, where $x > 1$. The length of the arc of the curve between the points on the curve where $x = 2$ and $x = 9$ is denoted by s.

(a) Show that $s = \int_2^9 \sqrt{\frac{x}{x-1}}\,dx$.

[3 marks]

(b) (i) Show that $\cosh^{-1} 3 = 2\ln(1+\sqrt{2})$.

[2 marks]

(ii) Use the substitution $x = \cosh^2\theta$ to show that

$$s = m\sqrt{2} + \ln(1+\sqrt{2})$$

where m is an integer.

[6 marks]

8.

a)

$$y = 2\sqrt{x-1} \quad \Rightarrow \quad arc\ length\ from\ x = 2\ to\ x = 9 = s = \int_2^9 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = 2(x-1)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = (x-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x-1}}$$

$$\Rightarrow \quad s = \int_2^9 \sqrt{1+\frac{1}{x-1}}\,dx = \int_2^9 \sqrt{1+\frac{1}{x-1}}\,dx = \int_2^9 \sqrt{\frac{x-1}{x-1}+\frac{1}{x-1}}\,dx = \boldsymbol{\int_2^9 \sqrt{\frac{x}{x-1}}\,dx}$$

b)

i.

Using the formula book definition:

$$\cosh^{-1} x = \ln\left\{x+\sqrt{x^2+1}\right\} \quad \Rightarrow \quad \cosh^{-1} 3 = \ln(3+\sqrt{8}) = \ln(3+2\sqrt{2})$$

OR, from first principles:

$$\cosh^{-1} 3 = x \quad \Rightarrow \quad 3 = \cosh x \quad \Rightarrow \quad 3 = \frac{e^x+e^{-x}}{2} \quad \Rightarrow \quad 6 = e^x + e^{-x}$$

$$\Rightarrow \quad e^{2x} - 6e^x + 1 = 0 \quad \Rightarrow \quad e^x = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2} \quad \Rightarrow \quad e^x = 3+2\sqrt{2} \quad \Rightarrow \quad x = \ln(3+2\sqrt{2})$$

$$(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2} \quad \Rightarrow \quad x = \ln\left((1+\sqrt{2})^2\right) = \mathbf{2\ln(1+\sqrt{2})}$$

ii.

$$x = \cosh^2\theta \quad \Longrightarrow \quad \frac{dx}{d\theta} = 2\cosh\theta\sinh\theta$$

$$\int_2^9 \sqrt{\frac{x}{x-1}}\,dx = \int_{\cosh^{-1}\sqrt{2}}^{\cosh^{-1}3} \sqrt{\frac{\cosh^2\theta}{\cosh^2\theta - 1}}\,2\cosh\theta\sinh\theta\ \ d\theta = \int_{\ln(1+\sqrt{2})}^{2\ln(1+\sqrt{2})} \sqrt{\frac{\cosh^2\theta}{\sinh^2\theta}}\,2\cosh\theta\sinh\theta\ \ d\theta$$

$$= \int_{\ln(1+\sqrt{2})}^{2\ln(1+\sqrt{2})} 2\cosh^2\theta\ \ d\theta = \int_{\ln(1+\sqrt{2})}^{2\ln(1+\sqrt{2})} \cosh 2\theta + 1\ d\theta = \left[\frac{\sinh 2\theta}{2} + \theta\right]_{\ln(1+\sqrt{2})}^{2\ln(1+\sqrt{2})}$$

$$= \left(\frac{\sinh\left(4\ln\left(1+\sqrt{2}\right)\right)}{2} + 2\ln\left(1+\sqrt{2}\right)\right) - \left(\frac{\sinh\left(2\ln\left(1+\sqrt{2}\right)\right)}{2} + \ln\left(1+\sqrt{2}\right)\right)$$

$$= \frac{\sinh\left(\ln\left(1+\sqrt{2}\right)^4\right)}{2} - \frac{\sinh\left(\ln\left(1+\sqrt{2}\right)^2\right)}{2} + \ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{4}\left(\left(1+\sqrt{2}\right)^4 - \left(1+\sqrt{2}\right)^{-4} - \left(\left(1+\sqrt{2}\right)^2 - \left(1+\sqrt{2}\right)^{-2}\right)\right) + \ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{4}\left(\left(\left(3+2\sqrt{2}\right)^2 - \frac{1}{\left(3+2\sqrt{2}\right)^2}\right) - \left(\left(3+2\sqrt{2}\right) - \frac{1}{3+2\sqrt{2}}\right)\right) + \ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{4}\left(\left(\left(17+12\sqrt{2}\right) - \frac{1}{17+12\sqrt{2}}\right) - \left(3+2\sqrt{2} - \frac{1}{3+2\sqrt{2}}\right)\right) + \ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{4}\left(\left(17+12\sqrt{2} - \left(17-12\sqrt{2}\right)\right) - \left(\left(3+2\sqrt{2}\right) - \left(3-2\sqrt{2}\right)\right)\right) + \ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{4}\left(24\sqrt{2} - 4\sqrt{2}\right) + \ln\left(1+\sqrt{2}\right) = \mathbf{5\sqrt{2} + \ln\left(1+\sqrt{2}\right)}$$