AQA MFP1 Further Pure 1 Further Mathematics 10 June 2014

Question Paper and Worked Solutions

Please note, this document represents my own solutions to the questions, is entirely unofficial and is not related to the mark scheme (which I have not seen). Therefore, while it should help you see how to do the questions, it won't include every valid method or give you a break down of the mark allocation. If you spot any errors, or think you have found a better solution, please email me so I can update it.

1 A curve passes through the point $(9, 6)$ and satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{2+\sqrt{x}}$$

Use a step-by-step method with a step length of 0.25 to estimate the value of y at $x = 9.5$. Give your answer to four decimal places.

[5 marks]

1.

$$y_{n+1} = y_n + h\,f(x_n) \quad where \quad x_{n+1} = x_n + h$$

$$h = 0.25 \quad x_0 = 9 \quad y_0 = 6$$

$$y_1 = y_0 + 0.25f(x_0) = 6 + 0.25 \times \left(\frac{1}{2+\sqrt{9}}\right) = 6.05$$

$$y_2 = y_1 + 0.25f(x_1) = 6.05 + 0.25 \times \left(\frac{1}{2+\sqrt{9.25}}\right) = \mathbf{6.0996\ to\ 4\ d.p.}$$

2 The quadratic equation

$$2x^2 + 8x + 1 = 0$$

has roots α and β.

(a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$.

[2 marks]

(b) (i) Find the value of $\alpha^2 + \beta^2$.

[2 marks]

(ii) Hence, or otherwise, show that $\alpha^4 + \beta^4 = \frac{449}{2}$.

[2 marks]

(c) Find a quadratic equation, with integer coefficients, which has roots

$$2\alpha^4 + \frac{1}{\beta^2} \text{ and } 2\beta^4 + \frac{1}{\alpha^2}$$

[5 marks]

2.

a)

$$\alpha + \beta = \Sigma\alpha = -\frac{b}{a} = -\frac{8}{2} = \mathbf{-4}$$

$$\alpha\beta = \frac{c}{a} = \frac{\mathbf{1}}{\mathbf{2}}$$

b)

i.

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-4)^2 - 2\left(\frac{1}{2}\right) = \mathbf{15}$$

ii.

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = (15)^2 - 2\left(\frac{1}{2}\right)^2 = \frac{\mathbf{449}}{\mathbf{2}}$$

c)

$$-\frac{b'}{a'} = \Sigma\alpha' = 2\alpha^4 + \frac{1}{\beta^2} + 2\beta^4 + \frac{1}{\alpha^2} = 2(\alpha^4 + \beta^4) + \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = 2\left(\frac{449}{2}\right) + \frac{15}{\left(\frac{1}{2}\right)^2} = 509$$

$$\frac{c'}{a'} = \alpha'\beta' = \left(2\alpha^4 + \frac{1}{\beta^2}\right)\left(2\beta^4 + \frac{1}{\alpha^2}\right) = 4(\alpha\beta)^4 + 2(\alpha^2 + \beta^2) + \frac{1}{(\alpha\beta)^2} = 4\left(\frac{1}{2}\right)^4 + 2(15) + \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{137}{4}$$

$$a' = 4 \quad \Rightarrow \quad b' = -509 \times 4 = -2036 \quad and \quad c' = 137 \quad \Rightarrow \quad \mathbf{4x^2 - 2036x + 137 = 0}$$

3 Use the formulae for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r^2$ to find the value of

$$\sum_{r=3}^{60} r^2(r-6)$$

[4 marks]

3.

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2 \qquad \sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{r=3}^{60} r^2(r-6) = \sum_{r=1}^{60} r^2(r-6) - \sum_{r=1}^{2} r^2(r-6) = \left\{\sum_{r=1}^{60} r^3 - 6\sum_{r=1}^{60} r^2\right\} - \left\{\sum_{r=1}^{2} r^3 - 6\sum_{r=1}^{2} r^2\right\}$$

$$= \left\{\left(\frac{60^2}{4}(60+1)^2\right) - 6\left(\frac{60}{6}(60+1)(2\times 60+1)\right)\right\} - \left\{\left(\frac{2^2}{4}(2+1)^2\right) - 6\left(\frac{2}{6}(2+1)(2\times 2+1)\right)\right\}$$

$$= \left\{\left(\frac{60^2}{4}(60+1)^2\right) - 6\left(\frac{60}{6}(60+1)(2\times 60+1)\right)\right\} - \left\{\left(\frac{2^2}{4}(2+1)^2\right) - 6\left(\frac{2}{6}(2+1)(2\times 2+1)\right)\right\}$$

$$= \{2906040\} - \{-21\} = \mathbf{2906061}$$

4 Find the complex number z such that

$$5\mathrm{i}z + 3z^* + 16 = 8\mathrm{i}$$

Give your answer in the form $a + b\mathrm{i}$, where a and b are real.

[6 marks]

4.

$$z = a + bi \quad \Rightarrow \quad z^* = a - bi$$

$$5i(a+bi) + 3(a-bi) + 16 = 8i \quad \Rightarrow \quad 5ai - 5b + 3a - 3bi + 16 = 8i$$

$$\Rightarrow \quad (3a - 5b + 16) + (5a - 3b)i = 8i \quad \Rightarrow \quad 3a - 5b + 16 = 0 \quad and \quad 5a - 3b = 8$$

$$15a - 25b = -80 \quad and \quad 15a - 9b = 24 \quad \Rightarrow \quad 16b = 104 \quad \Rightarrow \quad b = \frac{13}{2} \quad \Rightarrow \quad a = \frac{8 + 3\left(\frac{13}{2}\right)}{5} = \frac{11}{2}$$

$$\Rightarrow \quad \boldsymbol{z = \frac{11}{2} + \frac{13}{2}i}$$

5 A curve C has equation $y = x(x+3)$.

(a) Find the gradient of the line passing through the point $(-5, 10)$ and the point on C with x-coordinate $-5+h$. Give your answer in its simplest form.

[3 marks]

(b) Show how the answer to part **(a)** can be used to find the gradient of the curve C at the point $(-5, 10)$. State the value of this gradient.

[2 marks]

5.

a)

$$x = -5 + h \quad \Rightarrow \quad y = (-5+h)(-5+h+3) = (h-5)(h-2) = h^2 - 7h + 10$$

$$Gradient = \frac{(h^2 - 7h + 10) - (10)}{(-5+h) - (-5)} = \frac{h^2 - 7h}{h} = \boldsymbol{h - 7}$$

b)

$$Gradient\ at\ (-5,10) = \lim_{h \to 0}\left[Gradient\ between\ (-5,10)\ and\ \left(-5+h, f(-5+h)\right)\right] = \lim_{h \to 0}(h - 7) = \mathbf{-7}$$

6 A curve C has equation $y = \dfrac{1}{x(x+2)}$.

(a) Write down the equations of all the asymptotes of C. **[2 marks]**

(b) The curve C has exactly one stationary point. The x-coordinate of the stationary point is -1.

(i) Find the y-coordinate of the stationary point. **[1 mark]**

(ii) Sketch the curve C. **[2 marks]**

(c) Solve the inequality

$$\frac{1}{x(x+2)} \leqslant \frac{1}{8}$$

[5 marks]

6.

a)

Vertical asymptotes occur when $\boldsymbol{x = 0}$ or $x + 2 = 0 \quad \Rightarrow \quad \boldsymbol{x = -2}$

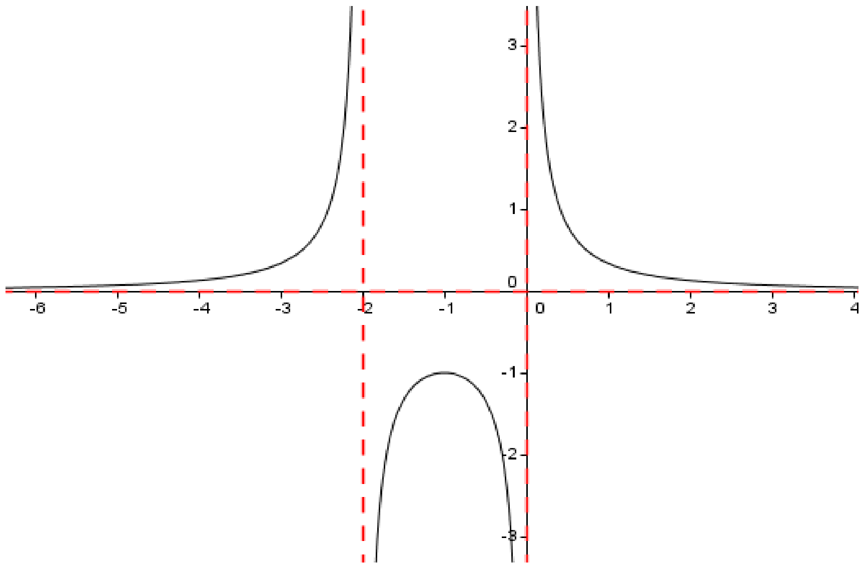
Horizontal asymptotes occur at: $y = \lim_{x\to\infty} \frac{1}{x(x+2)} = \lim_{x\to\infty} \frac{1}{x^2+2x} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1+\frac{2}{x}} = \frac{0}{1+0} = 0 \quad \Rightarrow \quad \boldsymbol{y = 0}$

b)

i.

$$x = -1 \quad \Rightarrow \quad y = \frac{1}{(-1)(-1+2)} = \boldsymbol{-1}$$

ii.



c)

$x < -2$	$-2 < x < 0$	$x > 0$
$8 \le x(x+2)$	$All\ values\ of\ x$	$8 \le x(x+2)$

$$x(x+2) = 8 \quad \Rightarrow \quad x^2 + 2x - 8 = 0 \quad \Rightarrow \quad (x+4)(x-2) = 0 \quad \Rightarrow \quad x = -4 \quad or \quad x = 2 \quad (critical\ values)$$

$$\frac{1}{x(x+2)} \le \frac{1}{8} \quad \Rightarrow \quad \boldsymbol{x \le -4 \quad or \quad -2 < x < 0 \quad or \quad x \ge 2}$$

7 (a) Write down the 2×2 matrix corresponding to each of the following transformations:

(i) a reflection in the line $y = -x$;

[1 mark]

(ii) a stretch parallel to the y-axis of scale factor 7.

[1 mark]

(b) Hence find the matrix corresponding to the combined transformation of a reflection in the line $y = -x$ followed by a stretch parallel to the y-axis of scale factor 7.

[2 marks]

(c) The matrix $\mathbf{A}$ is defined by $\mathbf{A} = \begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$.

(i) Show that $\mathbf{A}^2 = k\mathbf{I}$, where k is a constant and $\mathbf{I}$ is the 2×2 identity matrix.

[1 mark]

(ii) Show that the matrix $\mathbf{A}$ corresponds to a combination of an enlargement and a reflection. State the scale factor of the enlargement and state the equation of the line of reflection in the form $y = (\tan\theta)x$.

[5 marks]

7.
a)
i.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} \Rightarrow \quad ax + by = -y \quad and \quad cx + dy = -x \quad \Rightarrow \quad a = 0 \quad b = -1 \quad c = -1 \quad d = 0$$

$$\Rightarrow \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix}$$

ii.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 7y \end{bmatrix} \Rightarrow \quad ax + by = x \quad and \quad cx + dy = 7y \quad \Rightarrow \quad a = 1 \quad b = 0 \quad c = 0 \quad d = 7$$

$$\Rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{7} \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-7} & \mathbf{0} \end{bmatrix}$$

c)
i.

$$\begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}\begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{12I}$$

ii.
Since $A^2 = 12I$, A combines the reflection with an **enlargement of scale factor $\sqrt{12} = 2\sqrt{3}$**

$$\frac{A}{2\sqrt{3}} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \Rightarrow \theta = -75° \Rightarrow \boldsymbol{Reflection\ in\ the\ line\ y = (\tan(-75°))x}$$

Alternatives:

$$Since \tan(-75) = \tan(105): \quad \boldsymbol{Enlargement\ of\ SF\ 2\sqrt{3}\ and\ reflection\ in\ \tan(105)}$$

$$Since\ 12 = \pm 2\sqrt{3}, another\ alternative\ is: \quad \boldsymbol{Enlargement\ of\ SF - 2\sqrt{3}\ and\ reflection\ in\ \tan(15)}$$

8 (a) Find the general solution of the equation

$$\cos\left(\frac{5}{4}x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$$

giving your answer for x in terms of π.

[5 marks]

(b) Use your general solution to find the **sum** of all the solutions of the equation $\cos\left(\frac{5}{4}x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$ that lie in the interval $0 \leqslant x \leqslant 20\pi$. Give your answer in the form $k\pi$, stating the exact value of k.

[4 marks]

8.

a)

$$\cos\left(\frac{5}{4}x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \frac{5}{4}x - \frac{\pi}{3} = \pm\frac{\pi}{4} + 2n\pi$$

$$\Rightarrow \quad x = \frac{4}{5}\left(\frac{\pi}{3} \pm \frac{\pi}{4} + 2n\pi\right) = \frac{\pi}{5}\left(\frac{4 \pm 3}{3} + 8n\right) = \boldsymbol{\frac{\pi(24n+7)}{15}} \quad \boldsymbol{or} \quad \boldsymbol{\frac{\pi(24n+1)}{15}}$$

b)

To find first solutions in the range:

$$n = 0 \Rightarrow \frac{\pi}{15} \ and \ \frac{7\pi}{15}$$

To find last solutions in the range:

$$\frac{\pi(24n+7)}{15} < 20\pi \Rightarrow 24n + 7 < 300 \Rightarrow n < \frac{293}{24} \Rightarrow n < 12.20.. \Rightarrow n = 12$$

And:

$$\frac{\pi(24n+1)}{15} < 20\pi \Rightarrow 24n + 1 < 300 \Rightarrow n < \frac{299}{24} \Rightarrow n < 12.45.. \Rightarrow n = 12$$

Therefore last two solutions are:

$$\frac{\pi(24(12)+1)}{15} \ and \ \frac{\pi(24(12)+7)}{15} \Rightarrow \frac{289\pi}{15} \ and \ \frac{295\pi}{15}$$

Sum of each pair of solutions (one pair for each value of n in the range):

$$\frac{\pi(24n+1)}{15} + \frac{\pi(24n+7)}{15} = \frac{\pi(48n+8)}{15}$$

For n from 0 to 12:

$$\frac{48\pi}{15}\sum_{r=0}^{12} n + \frac{8\pi}{15}(12+1) = \frac{48\pi}{15}\left(\frac{12}{2}(12+1)\right) + \frac{108\pi}{15} = \frac{3744\pi}{15} + \frac{108\pi}{15} = \boldsymbol{\frac{3848\pi}{15}} \quad \boldsymbol{where\ k = \frac{3848}{15}}$$

9 An ellipse E has equation

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

(a) Sketch the ellipse E, showing the values of the intercepts on the coordinate axes.
[2 marks]

(b) Given that the line with equation $y=x+k$ intersects the ellipse E at two distinct points, show that $-5<k<5$.
[5 marks]

(c) The ellipse E is translated by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ to form another ellipse whose equation is $9x^2+16y^2+18x-64y=c$. Find the values of the constants a, b and c.
[5 marks]

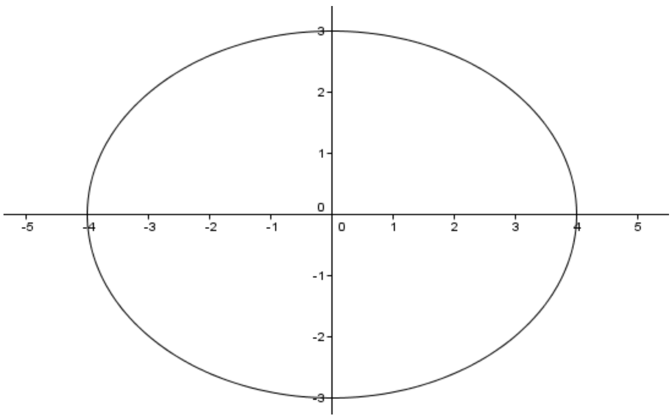
(d) **Hence** find an equation for each of the two tangents to the ellipse $9x^2+16y^2+18x-64y=c$ that are parallel to the line $y=x$.
[3 marks]

9.
a)

$$\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2=1$$

$$x=0 \Rightarrow y=\pm 3$$

$$y=0 \Rightarrow x=\pm 4$$



b)

$$y=x+k \Rightarrow \frac{x^2}{16}+\frac{(x+k)^2}{9}=1 \Rightarrow 9x^2+16(x+k)^2=144 \Rightarrow 25x^2+32kx+16k^2-144=0$$

$$\textit{Two distinct solutions} \Rightarrow b^2-4ac>0 \Rightarrow (32k)^2-4(25)(16k^2-144)>0$$

$$\Rightarrow 16k^2-25(k^2-9)>0 \Rightarrow -9k^2+225>0 \Rightarrow k^2<25 \Rightarrow \boldsymbol{-5<k<5}$$

c)

$$9x^2+16y^2+18x-64y=c \Rightarrow 9(x^2+2x)+16(y^2-4y)=c$$

$$\Rightarrow 9((x+1)^2-1)+16((y-2)^2-4)=c \Rightarrow 9(x+1)^2-9+16(y-2)^2-64=c$$

$$9(x+1)^2+16(y-2)^2=c+73 \Rightarrow \frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=\frac{c+73}{144}$$

$$\boldsymbol{a=-1} \quad \boldsymbol{b=2} \quad \frac{c+73}{144}=1 \Rightarrow \boldsymbol{c=71}$$

d)

$$\frac{x^2}{16}+\frac{y^2}{9}=1 \quad \textit{is tangent to the lines} \quad y=x+5 \quad \textit{and} \quad y=x-5$$

Translating these lines by vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ gives tangent lines to the new ellipse:

$$y-2=(x+1)+5 \Rightarrow \boldsymbol{y=x+8} \qquad \textit{and} \qquad y-2=(x+1)-5 \Rightarrow \boldsymbol{y=x-2}$$