Simultaneous Equations: Line & Curve

Although elimination is often the most efficient method for ‘nice’ linear simultaneous equations, more complicated equations require substitution.

For equations involving curves, there are often multiple solutions:

- Circle & line: 0, 1 or 2 solutions
- Quadratic & line: 0, 1 or 2 solutions
- Quadratic & quadratic: 0, 1 or 2 solutions
  (you won’t see these at GCSE)

Don’t mix up the x and y coordinates: (1,2) and (6,7) are not the same points as (1,7) and (6,2)

Example

Solve simultaneously:

\[ x^2 + y^2 = 5 \]
\[ y = 3x + 1 \]

Substitute for y, from the linear into the quadratic:

\[ x^2 + (3x + 1)^2 = 5 \]

Simplify, rearrange and solve the resulting quadratic:

\[ x^2 + 9x^2 + 6x + 1 = 5 \]
\[ 10x^2 + 6x - 4 = 0 \Rightarrow 5x^2 + 3x - 2 = 0 \]
\[ (5x - 2)(x + 1) = 0 \Rightarrow x = 0.4 \text{ or } x = -1 \]

Substitute each x value back into the linear equation:

- \( x = 0.4 \Rightarrow y = 3(0.4) + 1 = 2.2 \Rightarrow (0.4, 2.2) \)
- \( x = -1 \Rightarrow y = 3(-1) + 1 = -2 \Rightarrow (-1, -2) \)

The circle shown has equation \( x^2 + y^2 = 25 \). Find any points of intersection with the lines below. Which never crosses the circle, and which is a tangent?

1) \( x = -4 \)
2) \( 4y = -3x \)
3) \( x + y = 8 \)
4) \( 3x + 4y = 25 \)
Simultaneous Equations: Line & Curve SOLUTIONS

1)  
\[ x = -4 \implies (-4)^2 + y^2 = 25 \]
\[ 16 + y^2 = 25 \]
\[ y^2 = 9 \]
\[ y = 3 \text{ or } y = -3 \]

Substitute back to find corresponding values of \( x \): (note: \( x = -4 \) anywhere on the line!)

\[ y = 3 \implies x = -4 \implies (-4,3) \]
\[ y = -3 \implies x = -4 \implies (-4,-3) \]

The line is vertical, so the crossing points are symmetrical about the \( x \) axis.

2.  
\[ y = -\frac{3}{4}x \implies x^2 + \left(-\frac{3}{4}x\right)^2 = 25 \]
\[ x^2 + \frac{9}{16}x^2 = 25 \]
\[ \frac{25}{16}x^2 = 25 \]
\[ x^2 = 16 \]
\[ x = 4 \text{ or } x = -4 \]

Substitute back to find corresponding values of \( y \):

\[ x = 4 \implies y = -\frac{3}{4}(4) = -3 \implies (4,-3) \]
\[ x = -4 \implies y = -\frac{3}{4}(-4) = 3 \implies (-4,3) \]

The line goes through the origin, so each point is a 180° rotation about the origin from the other.
3) \[ x = 8 - y \Rightarrow (8 - y)^2 + y^2 = 25 \]
\[ 64 - 16y + y^2 + y^2 = 25 \]
\[ 2y^2 - 16y + 39 = 0 \]

\[ b^2 - 4ac = (-16)^2 - 4(2)(39) = -56 < 0 \]

\[ b^2 - 4ac < 0 \Rightarrow \text{No Solutions} \]

The line never touches the circle.
This is the geometric interpretation of finding no solutions to the resulting quadratic equation.

4) \[ y = \frac{25 - 3x}{4} \Rightarrow x^2 + \left(\frac{25 - 3x}{4}\right)^2 = 25 \]
\[ x^2 + \frac{(25 - 3x)^2}{16} = 25 \]
\[ 16x^2 + (25 - 3x)^2 = 400 \]
\[ 16x^2 + 625 - 150x + 9x^2 = 400 \]
\[ 25x^2 - 150x + 225 = 0 \]
\[ x^2 - 6x + 9 = 0 \]
\[ (x - 3)(x - 3) = 0 \]
\[ x = 3 \]

Substitute back to find corresponding value of \( y \):
\[ x = 3 \Rightarrow y = \frac{25 - 3(3)}{4} = 4 \Rightarrow (3, 4) \]

The line is a tangent line, only meeting the circle at one point.
This is the geometric interpretation of a repeated root (exactly one solution) when solving the quadratic equation.