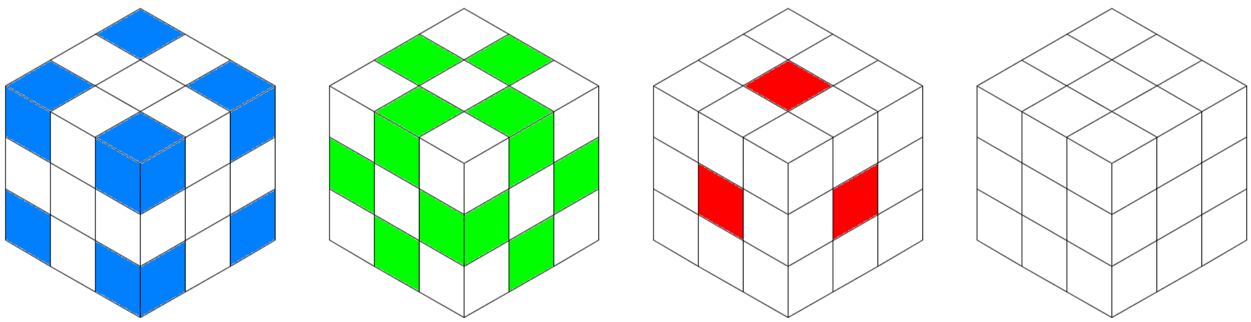
Painted Cubes Investigation

A number of small cubes are to be assembled into a larger solid cube. Once built, the entire outer surface of this large cube is to be painted. Some of the small cubes will end up with none of their faces painted, some with just one, some with two and some with three faces painted.

Your challenge is to find a way of predicting, for any given size of large cube, how many of each type there will be.

Here are some diagrams of the 3 by 3 by 3 cube which you may find useful:



You will have access to isometric paper should you wish to draw out cubes, and squared paper for your calculations, tables, etc.

You are encouraged to work collaboratively on this project – while simply copying another person's ideas would be counter-productive (you've no way of knowing if they're correct, and you won't be able to explain how you thought of them), discussing ideas and sharing your thoughts with other people in your group will be beneficial to all of you.

When you come up with a conjecture (an idea about how a pattern progresses, or a formula you think will predict, for instance, the number of small cubes with 2 faces painted), write it – as clearly as you can – in one of the boxes on your conjectures sheet. Don't get rid of conjectures that turn out to be wrong – they are all part of the problem solving process, and often incorrect ideas allow us to rule out certain lines of thought and help us to see what we should be focusing on.

Painted Cubes Conjectures

A conjecture is an idea about how something works, or a possible rule or formula that may predict something.

They may turn out to be correct or not – that doesn't matter. The importance of conjectures is that they help you to direct your thinking, rule out certain lines of thought and give you clear goals for your investigation.

Record every conjecture here, as clearly as you can, and fill in the extra details about it as you continue with your investigation:

Conjecture 1

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

Incorrect Partially correct Correct

Conjecture 2

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

Incorrect Partially correct Correct

Conjecture

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

Incorrect Partially correct Correct

Conjecture

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

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Conjecture

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

Incorrect Partially correct Correct

Painted Cubes Investigation SOLUTIONS

n^{th} term rules:

Painted faces:	0	1	2	3
$n \times n \times n$	$(n-2)^3$	$6(n-2)^2$	$12(n-2)$	8

Geometric explanations:

0 painted faces:
These form a cube within the larger cube whose side length is two shorter than that of the larger cube (since the outer layer of cubes in all directions have at least one face painted). The volume of this cube must be the side length cubed: $(n-2)^3$.

1 painted face:
These are in the outer layer, but cannot be at the edge or corners. They form squares within each face of the cube. Each square has side length two less than the length of the original cube due to the edges around, and there are six faces: $6(n-2)^2$.

2 painted faces:
These are the edge cubes. Each edge contains two fewer of these cubes than the dimensions of the original cube due the corner cubes at each end, and there are 12 edges: $12(n-2)$.

3 painted faces:
These only occur in the corners, of which there are 8. This is completely independent of the size of the cube (providing it is at least $2 \times 2 \times 2$): 8.

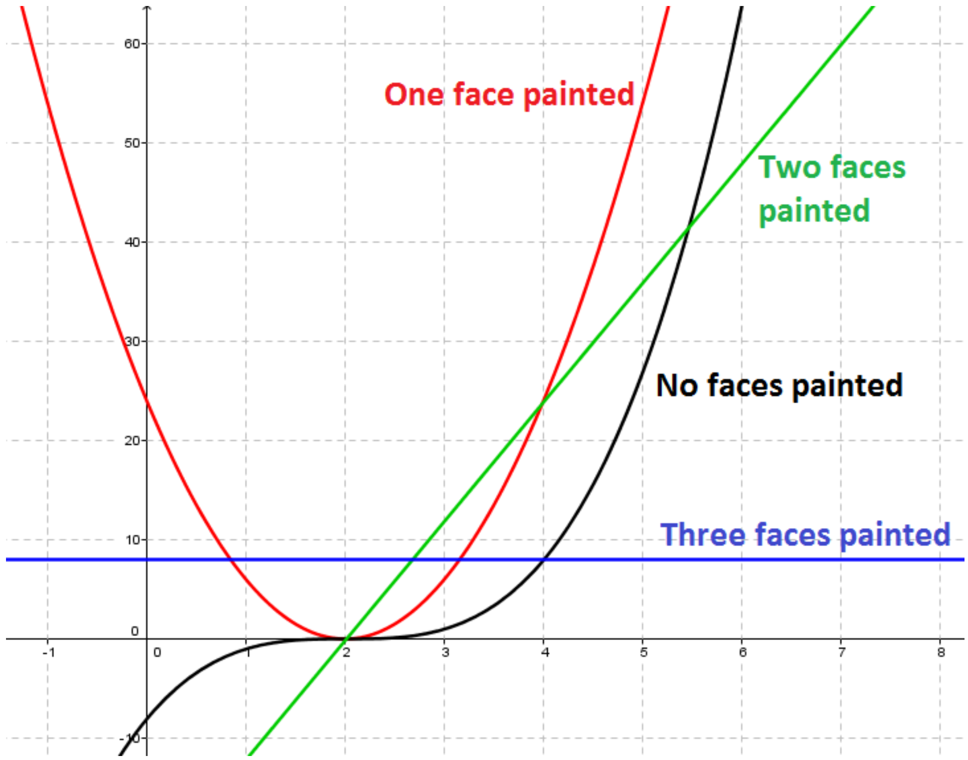
Important cubes:

$1 \times 1 \times 1$ does not fit the rules, since it is only made of a single cube (with 6 faces painted).
$2 \times 2 \times 2$ has more corner cubes (3 faces painted) than any others. In fact, it is all corners.
$3 \times 3 \times 3$ is the first cube with more edge cubes (2 faces painted) than corner cubes.
$4 \times 4 \times 4$ is the first cube to have at least as many face cubes (1 face painted) as edge cubes.
$8 \times 8 \times 8$ is the first cube to have at least as many inner cubes (0 faces painted) as face cubes.

$100 \times 100 \times 100$ has 941192 inner cubes, 57624 face cubes, 1176 edge cubes and 8 corner cubes, meaning over 94% of the cubes have no faces painted.

Painted Cubes Investigation GRAPHS

Since the functions are successively constant, linear, quadratic and cubic, the graphs, in the long run, will show why non-painted cubes become more common than any of the others:



And with a slightly different scale, to show all the crossing points: