The mathematics of circular motion

In the special case of uniform circular motion, while the displacement, velocity and acceleration are constantly changing, the *distance*, *speed* and *magnitude of acceleration* are all constant (it is only the direction of the vectors which changes). By examining the situation from the point of view of two-dimensional kinematics, we can find both the size and the direction of these vectors. Follow this exam question to see the mathematical link between uniform circular motion and kinematics: (*AQA Mechanics 2B, Jun '12)*

4 A particle moves on a horizontal plane, in which the unit vectors **i** and **j** are perpendicular.

At time t, the particle's position vector, **r**, is given by

$$\mathbf{r} = 4\cos 3t\,\mathbf{i} - 4\sin 3t\,\mathbf{j}$$

(a) Prove that the particle is moving on a circle, which has its centre at the origin. *(2 marks)*

Note that the position vector is given in $\boldsymbol{i}, \boldsymbol{j}$ form, with each component in terms of the time t.

To prove that the particle is performing circular motion about the origin, it is sufficient to show that the *distance* from the origin is constant. Since $\boldsymbol{r}$ is the displacement vector, the distance is given by $|\boldsymbol{r}|$:

$$|\boldsymbol{r}| = \sqrt{(4\cos 3t)^2 + (-4\sin 3t)^2} = 4\sqrt{\cos^2 3t + \sin^2 3t} = 4$$

$$Constant \quad \Rightarrow \quad Exhibiting\ circular\ motion\ with\ centre\ the\ origin.$$

By considering how the $\boldsymbol{i}$ and $\boldsymbol{j}$ components vary as $3t$ varies between 0 and 2π, we can get an impression not just of position, but also of velocity. Eg, while $0 \le 3t \le \frac{\pi}{2}$ the particle is in the fourth quadrant – in the $\boldsymbol{i}$ direction the position is positive but decreasing slowly, and in the $\boldsymbol{j}$ direction the position is negative and decreasing rapidly. In other words, the particle is travelling clockwise around a circle of radius 4 about the origin, starting from the point $4\boldsymbol{i} + 0\boldsymbol{j}$.

(b) Find an expression for the velocity of the particle at time t. *(2 marks)*

Recall that the velocity can be found from displacement using the result $\boldsymbol{v} = \frac{dr}{dt}$, and when $\boldsymbol{v}$ and $\boldsymbol{r}$ are vectors, this means differentiating the $\boldsymbol{i}$ and $\boldsymbol{j}$ components separately, with respect to time (these components are perpendicular, so do not directly affect one another).

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -12\sin 3t\,\boldsymbol{i} - 12\cos 3t\,\boldsymbol{j}$$

By considering the components of velocity at various points, just as we did for displacement, we can get an idea of the velocity of the particle. For instance, when $\frac{\pi}{2} \le 3t \le \pi$, the particle is moving left (quickly at first, but with decreasing speed) and up (slowly at first, but with increasing speed). Note also that, using the same method as in part a), we could use Pythagoras to show that the magnitude of velocity (the speed) is constant. Therefore while direction is constantly changing, the particle is travelling at a constant speed. Since the acceleration of the particle is affecting both components of velocity, it is harder to see how this is changing directly from this expression, but it will become clearer once we find the acceleration vector. Note, however, that as the $\boldsymbol{i}$ component of velocity increases, the $\boldsymbol{j}$ component decreases and vice versa. We should find that the acceleration of the particle is of constant magnitude even though its direction is constantly changing.

(c) Find an expression for the acceleration of the particle at time t. *(2 marks)*

Recall that the acceleration vector can be found from velocity using the result $\boldsymbol{a} = \frac{dv}{dt}$.

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -36\cos 3t\,\boldsymbol{i} + 36\sin 3t\,\boldsymbol{j}$$

Note firstly that it would be easy to show, as in part a), that acceleration is constant in magnitude (if not direction). Also note the similarity of this expression to that of displacement. The only difference is the signs (negative $\boldsymbol{i}$ and positive $\boldsymbol{j}$ in this case) and the magnitude. So the acceleration is clearly directly related to the displacement of the particle. This is a key feature of circular motion.

(d) The acceleration of the particle can be written as

$$\mathbf{a} = k\mathbf{r}$$

where k is a constant.

Find the value of k. *(2 marks)*

This is a straightforward result to prove, but it is crucial to the idea of circular moti on.

$$\boldsymbol{a} = -36\cos 3t\,\boldsymbol{i} + 36\sin 3t\,\boldsymbol{j} = -9(4\cos 3t\,\boldsymbol{i} - 4\sin 3t\,\boldsymbol{j}) = -9\boldsymbol{r}$$

Since the acceleration vector is a scalar multiple of the displacement vector, the two vectors are parallel. This means acceleration *always* acts along the same line as the displacement of the particle from the origin (that is, along the radius of the circular motion).

(e) State the direction of the acceleration of the particle. *(1 mark)*

$$Towards\ the\ centre\ of\ the\ circle\ (the\ origin)$$

Note the negative sign – this tells us that the acceleration acts in the opposite direction to displacement. Displacement measures the position of the particle relative to the origin, and so it always points from the origin to the particle. Therefore acceleration points from the particle to the origin. This fits in with the whole concept of centripetal force and centripetal acceleration. Since the velocity is constantly changing direction, the acceleration vector must be constantly changing direction, and since acceleration is always pointing radially (towards the centre), it cannot affect the magnitude of velocity which always points tangentially (at right angles to the radial direction).

In general, uniform circular motion is, by definition, motion at a constant angular velocity (and therefore constant speed) along a path a fixed distance from a given point.

At a distance r and angular velocity ω:

$$\boldsymbol{r} = \begin{bmatrix} r\cos\omega t \\ r\sin\omega t \end{bmatrix} \Rightarrow \boldsymbol{v} = \begin{bmatrix} -\omega r\sin\omega t \\ \omega r\cos\omega t \end{bmatrix} \Rightarrow \boldsymbol{a} = \begin{bmatrix} -\omega^2 r\cos\omega t \\ -\omega^2 r\sin\omega t \end{bmatrix}$$

Which gives us these general results:

Vector	Displacement $\boldsymbol{r}$	Velocity $\boldsymbol{v}$	Acceleration $\boldsymbol{a}$
Magnitude	$\boldsymbol{r} = r$ (radius)	$\boldsymbol{v} = \omega r$ (angular speed × radius)	$\boldsymbol{a} = \omega^2 r$ (angular speed ^{2} × radius)
Direction	Radial (*away* from centre)	Tangential (along a tangent)	Radial (*towards* the centre)