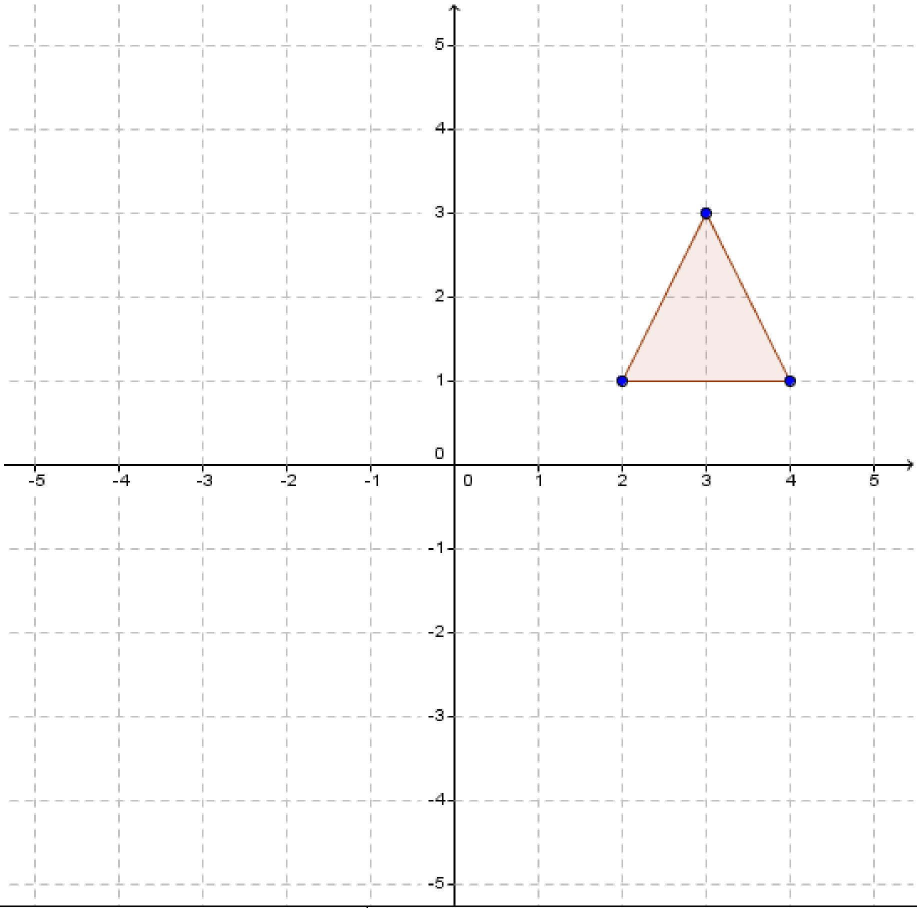
Matrix Transformations



The position of the corners of this triangle are described by the vectors:

$$\begin{bmatrix}2\\1\end{bmatrix} \quad \& \quad \begin{bmatrix}4\\1\end{bmatrix} \quad \& \quad \begin{bmatrix}3\\3\end{bmatrix}$$

Use each of the matrices below to transform these corners.

In each case, draw the new shape (the 'image') on the grid opposite and describe the transformation in words.

The first one is done for you.

$A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$First\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\times 2+0\times 1\\0\times 2+1\times 1\end{bmatrix} = \begin{bmatrix}\mathbf{2}\\\mathbf{1}\end{bmatrix}$ $Second\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}\mathbf{4}\\\mathbf{1}\end{bmatrix}$ $Third\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}\mathbf{3}\\\mathbf{3}\end{bmatrix}$ *Transformation: No change.*
$B = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$	*Transformation:*
$C = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	*Transformation:*
$D = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	*Transformation:*

How to multiply a matrix by a position vector to find the image of a transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

How to multiply a matrix by a series of position vectors all at once:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 & ax_3 + by_3 \\ cx_1 + dy_1 & cx_2 + dy_2 & cx_3 + dy_3 \end{bmatrix}$$

The rows of the first matrix define the transformation being applied.
Each column of the second matrix represents a point of the original shape.
Each column of the resulting matrix represents the corresponding point of the image.

How to multiply a matrix by a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

Two transformation matrices can be combined to form a new transformation matrix.
Since applying a matrix to a position vector involves putting the matrix on the left, the left-most matrix represents the most recent transformation.

Eg:
A stretch of scale factor 2, followed by a reflection in the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

General multiplication of matrices:

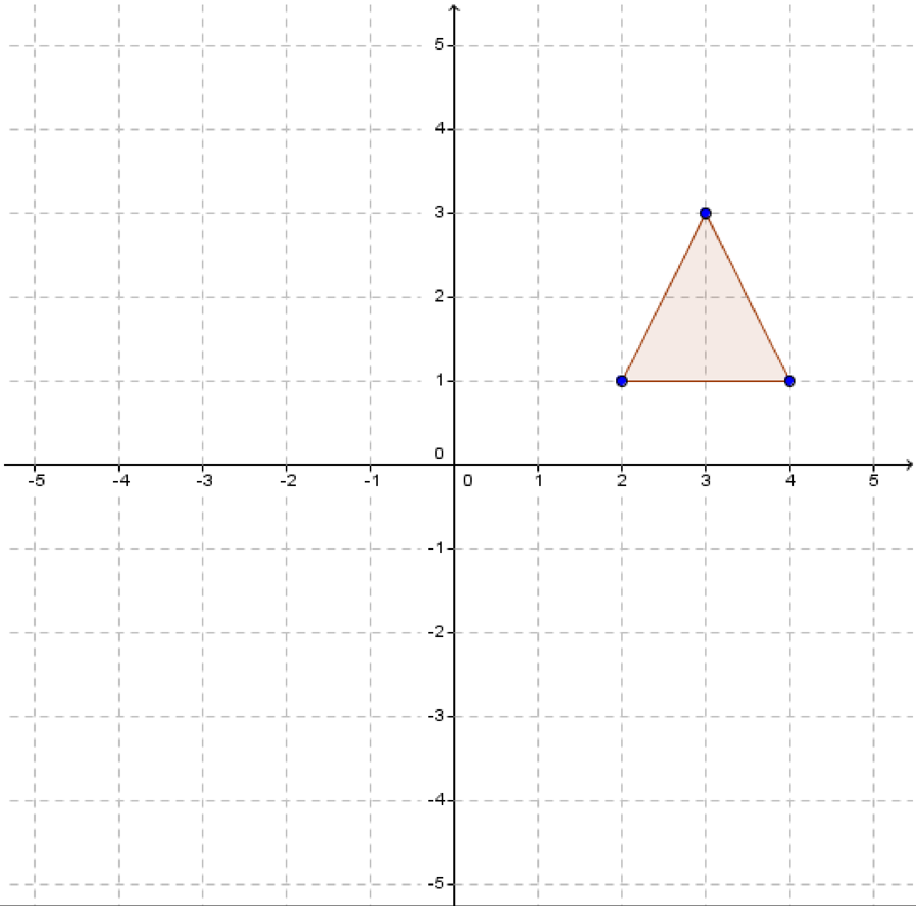
Each row of the first matrix is combined with each column of the second matrix. It can be helpful to think of the rows of the first matrix as defining a function whose input data is contained in the columns of the second matrix. You can have as many different functions as you like (rows in the first matrix), and as many sets of data as you like (columns in the second matrix) but the number of inputs each function takes is equal to the columns of the first and also to the rows of the second, so there must be the same number of each.

For instance:

$$\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 a + f_2 b \\ g_1 a + g_2 b \\ h_1 a + h_2 b \end{bmatrix}$$

$$(\mathbf{3}\ by\ 2) \times (2\ by\ \mathbf{1}) = (\mathbf{3}\ by\ \mathbf{1})$$

Matrix Transformations



The position of the corners of this triangle are described by the vectors:

$$\begin{bmatrix}2\\1\end{bmatrix} \quad \& \quad \begin{bmatrix}4\\1\end{bmatrix} \quad \& \quad \begin{bmatrix}3\\3\end{bmatrix}$$

Use each of the matrices below to transform these corners.

In each case, draw the new shape (the 'image') on the grid opposite and describe the transformation in words.

The first one is done for you.

$E = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$	$First\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\times 2+1\times 1\\1\times 2+0\times 1\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\\mathbf{2}\end{bmatrix}$ $Second\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\\mathbf{4}\end{bmatrix}$ $Third\ point$: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}\mathbf{3}\\\mathbf{3}\end{bmatrix}$ *Transformation: x and y values swapped, so: Reflection in $y = x$*
$F = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$	*Transformation:*
$G = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	*Transformation:*
$H = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$	*Transformation:*

How to multiply a matrix by a position vector to find the image of a transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

How to multiply a matrix by a series of position vectors all at once:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 & ax_3 + by_3 \\ cx_1 + dy_1 & cx_2 + dy_2 & cx_3 + dy_3 \end{bmatrix}$$

The rows of the first matrix define the transformation being applied.
Each column of the second matrix represents a point of the original shape.
Each column of the resulting matrix represents the corresponding point of the image.

How to multiply a matrix by a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

Two transformation matrices can be combined to form a new transformation matrix.
Since applying a matrix to a position vector involves putting the matrix on the left, the left-most matrix represents the most recent transformation.

Eg:
A stretch of scale factor 2, followed by a reflection in the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

General multiplication of matrices:

Each row of the first matrix is combined with each column of the second matrix. It can be helpful to think of the rows of the first matrix as defining a function whose input data is contained in the columns of the second matrix. You can have as many different functions as you like (rows in the first matrix), and as many sets of data as you like (columns in the second matrix) but the number of inputs each function takes is equal to the columns of the first and also to the rows of the second, so there must be the same number of each.

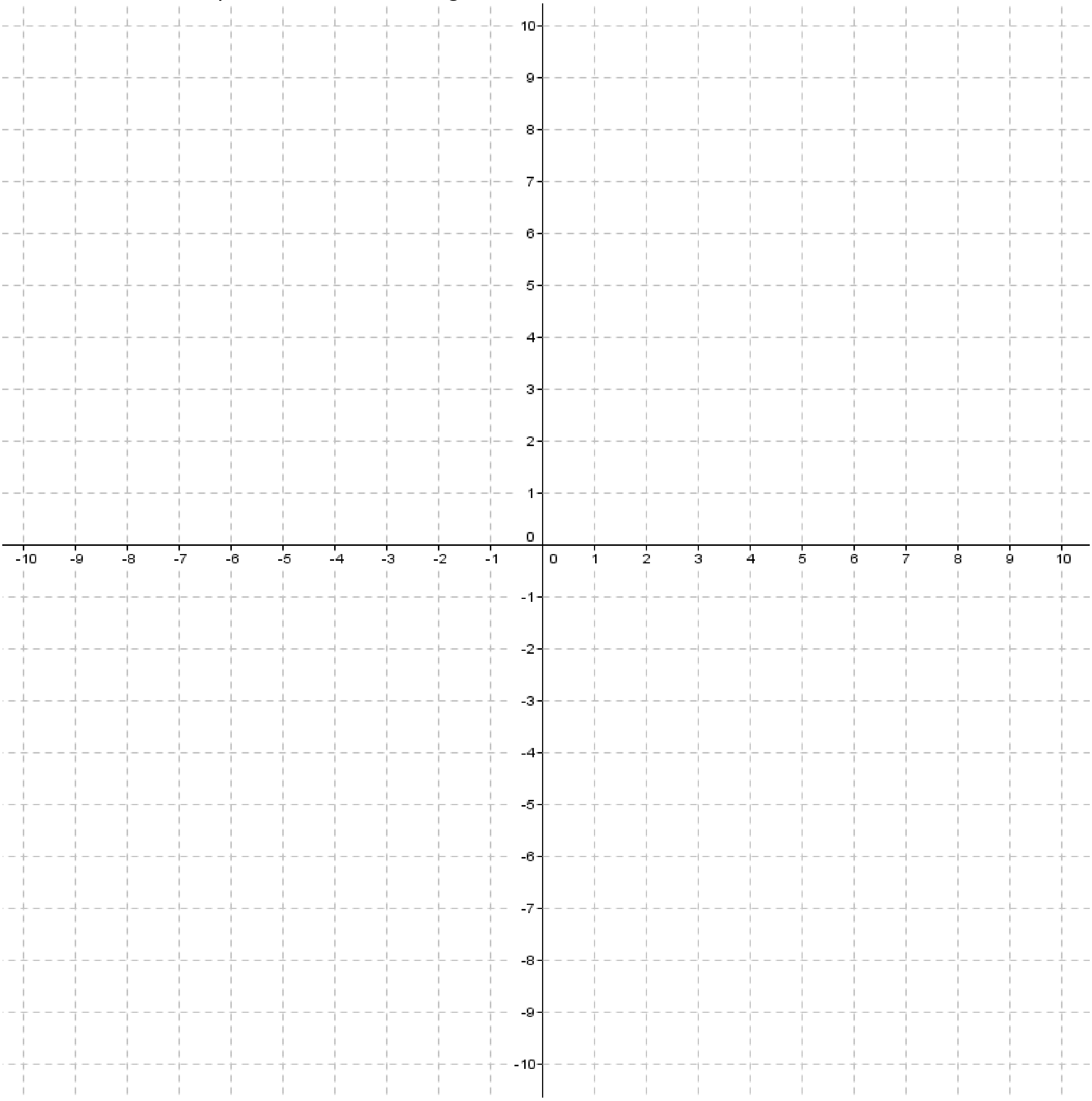
For instance:

$$\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 a + f_2 b \\ g_1 a + g_2 b \\ h_1 a + h_2 b \end{bmatrix}$$

$$(\mathbf{3}\ by\ 2) \times (2\ by\ \mathbf{1}) = (\mathbf{3}\ by\ \mathbf{1})$$

Do It Yourself

1. Construct a shape on the grid below by connecting points with lines.
It can be more complicated than a triangle, but shouldn't have more than around 5 or 6 sides.



2. Write a matrix with the position vectors of the corners as the columns.
There will be 2 rows, for the x and y coordinates, and one column for each corner.

3. Design a $2\ by\ 2$ transformation matrix.
This can be a combination of matrices you have already seen, or you can invent a new one.

4. Apply your transformation to the matrix of points to generate an output matrix.
This matrix will be the same size as your original matrix, with one column for each corner.

5. Plot the points described by your matrix columns and draw the new shape.
For complicated transformations, label your points A, B, … and then new points as A', B', etc.

6. Try to describe your transformation in terms of stretches, rotations and reflections.
If you built your matrix by combining other transformation matrices, you should be able to see how your transformation is equivalent to applying each of the separate transformations.

7. Apply your transformation matrix to the general position vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to find the image $\begin{bmatrix} x' \\ y' \end{bmatrix}$.
Use the following format:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{array}{c} \boldsymbol{x' = ax + by} \\ \boldsymbol{and} \\ \boldsymbol{y' = cx + dy} \end{array}$$

Solutions

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Identity$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad Reflection\ in\ y - axis$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Reflection\ in\ x - axis$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad Rotation\ by\ 180°$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Reflection\ in\ y = x$$

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad Rotation\ by\ 90°\ anticlockwise$$

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad Rotation\ by\ 90°\ clockwise$$

$$H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad Reflection\ in\ y = -x$$