AQA Level 2 Certificate in Further Mathematics

The Need-To-Know Booklet

Further Maths

Level 2

Every fact, result and method that you need to know

The Need-To-Know Book for Further Maths Level 2

Everything you need to know for Further Maths Level 2

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use for this qualification.
Note: The Further Maths Level 2 course is intended for those who have achieved, or are expecting to achieve, an A or A grade in GCSE Mathematics. As such, a thorough knowledge of the GCSE course is required as a prerequisite to this course.*

Contents

Section 1 – Number

Fractions

When **adding or subtracting fractions**, write over a common denominator, then add or subtract the numerators. Answers should be simplified where possible.

Note: If adding fractions in mixed number form, it is often easiest to add the whole number and fraction parts separately.

Eg:

$$4\frac{4}{5} + 5\frac{3}{7} = (4+5) + \left(\frac{4}{5} + \frac{3}{7}\right) = 9 + \left(\frac{28}{35} + \frac{15}{35}\right) = 9 + \left(\frac{43}{35}\right) = 9 + \left(1\frac{8}{35}\right) = \mathbf{10\frac{8}{35}}$$

When **multiplying fractions**, there is no need to find a common denominator. Simply multiply the numerators and multiply the denominators.

Note: If multiplying fractions in mixed number form, first write as improper fractions.

Note: For large numbers it can help to identify common factors in the numerators and denominators and pre-cancel before multiplying together.

Eg:

$$\frac{8}{15} \times \frac{9}{16} = \frac{1}{5} \times \frac{3}{2} = \mathbf{\frac{3}{10}}$$

Here the 8 and 16 were both divided by 8, and the 9 and 15 were both divided by 3.

When **dividing fractions**, multiply by the reciprocal of the fraction you want to divide by.

Note: If dividing fractions in mixed number form, first write as improper fractions.

When finding a **percentage of an amount**, convert to a decimal and multiply.

Eg:

$$26\% \ of \ £400 = 0.26 \times 400 = £\mathbf{104}$$

To **increase or decrease by a percentage**, multiply by the appropriate decimal.

Eg:

Increase 50kg by 15%: $1.15 \times 50 = \mathbf{57.5kg}$.

Decrease 80cm by 8%: $0.92 \times 80 = \mathbf{73.6cm}$.

To **reverse a percentage change**, divide by the relevant decimal multiplier.

Eg: A coat is marked down 30% in a sale. If the sale price is £43.40, what was the original price?

$$£43.4 \div 0.7 = £\mathbf{62}$$

To apply **compound interest** for n years, raise the decimal multiplier to the power n.

Eg: James borrows £2500 at an annual interest rate of 17%. How much will he owe after 8 years?

$$£2500 \times 1.17^8 = £\mathbf{8778.63}$$

To **share an amount in a ratio**, find the total number of parts, then scale up to the required total.

Eg: Julie and Kate work 8 hours and 14 hours respectively each week at a restaurant, and any tips received are shared in this ratio. One week they receive £55 in tips. How much does Kate receive?

$$8 : 14 \quad \Leftrightarrow \quad 4 : 7$$

Julie		Kate	Total
4	:	7	11
$\times 5$		$\times 5$	$\times 5$
20	**:**	**35**	**55**

$$\boldsymbol{Kate\ receives}\ £\mathbf{35}$$

Ratios can be **combined** by making the parts equal. This can be done by forming equivalent ratios.

Eg: There is twice as much sugar as flour in a recipe, and 2 parts butter to 3 parts sugar. What is the ratio of sugar to flour to butter?

$S : F \;=\; 2 : 1 \qquad B : S \;=\; 2 : 3$

$S : F \;=\; 6 : 3 \qquad B : S \;=\; 4 : 6$

$F : S \;=\; 3 : 6 \qquad S : B \;=\; 6 : 4$

$$F : S : B \;=\; 3 : 6 : 4$$

$$S : F : B \;=\; \mathbf{6 : 3 : 4}$$

Surds

A **surd** is an **irrational number involving a root**. An irrational number is one that cannot be written as a fraction with whole numbers forming the numerator and denominator.

Eg: $\sqrt{5}$ is a surd. $\sqrt{16} = 4$ so it is not a surd. Neither is $\sqrt{\frac{25}{4}}$ as $\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$.

Multiplication or **division** within a root can be brought outside the root. In general:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \quad and \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Note: Remember that addition does *not* work in the same way. For instance, $\sqrt{9+16} \neq \sqrt{9} + \sqrt{16}$.

Eg: A right-angled triangle has sides of $12cm$ and $8cm$. Find the possible lengths of the third side.

$$By\ Pythagoras'\ theorem:$$

$$Option\ 1\ (missing\ side\ hypotenuse):\quad \sqrt{12^2 + 8^2} = \sqrt{208} = \sqrt{16}\sqrt{13} = \mathbf{4\sqrt{13}}$$

$$Option\ 2\ (missing\ side\ not\ hypotenuse):\quad \sqrt{12^2 - 8^2} = \sqrt{80} = \sqrt{16}\sqrt{5} = \mathbf{4\sqrt{5}}$$

If a fraction has a **surd in the denominator**, it is often useful to rewrite the fraction in such a way as to have only rational numbers in the denominator. This is called **rationalising the denominator**. This is done, in simple cases, by simply multiplying top and bottom by the surd in the denominator:

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Eg: Simplify the following expression as far as possible: $\frac{5}{\sqrt{8}} - \sqrt{18} + \frac{\sqrt{80}}{\sqrt{5}}$

$$\frac{5}{\sqrt{8}} - \sqrt{18} + \frac{\sqrt{160}}{\sqrt{5}} = \frac{5\sqrt{8}}{8} - 3\sqrt{2} + \frac{4\sqrt{10}\sqrt{5}}{5} = \frac{10\sqrt{2}}{8} - 3\sqrt{2} + \frac{4\sqrt{2}\sqrt{5}\sqrt{5}}{5} = \frac{5\sqrt{2}}{4} - 3\sqrt{2} + 4\sqrt{2} = \mathbf{\frac{9\sqrt{2}}{4}}$$

To **rationalise the denominator** of a more complex fraction, use the **difference of two squares**:

$$\frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})} = \frac{ab - a\sqrt{c}}{b^2 - c}$$

Eg:

$$\frac{6}{3-2\sqrt{5}} = \frac{6(3+2\sqrt{5})}{(3-2\sqrt{5})(3+2\sqrt{5})} = \frac{18+12\sqrt{5}}{9-20} = \frac{18+12\sqrt{5}}{-11} = -\mathbf{\frac{6(3+2\sqrt{5})}{11}}$$

Section 2 – Algebra

Expressions

To **multiply out brackets**, multiply every term within the bracket by the multiplier outside:

$$a(b + c) = ab + ac$$

Note: You will also need to be familiar with rules of indices when dealing with these questions.

Eg: Expand the expression $3x^4y(2xy - 5x^3)$

$$3x^4y(2xy - 5x^3) = 3x^4y \times 2xy - 3x^4y \times 5x^3 = \mathbf{6x^5y^2 - 15x^7y}$$

When **simplifying** an algebraic expression where terms are added, you can **collect like terms**.

Eg: Simplify $3x(2 - 5x) - 7(x - 5)$

$$3x(2 - 5x) - 7(x - 5) = 6x - 15x^2 - (7x - 35) = 6x - 15x^2 - 7x + 35 = \mathbf{35 - x - 15x^2}$$

To **multiply brackets together**, every term in each bracket must be multiplied:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Note: To multiply out more than two brackets, it is usually easiest to take one pair at a time.

Eg: Multiply out $(3x - 1)(x + 4)(2 - x)$

$$(3x - 1)(x + 4)(2 - x) = (3x - 1)(2x - x^2 + 8 - 4x) = (3x - 1)(8 - 2x - x^2)$$

$$= 24x - 6x^2 - 3x^3 - 8 + 2x + x^2 = \mathbf{-8 + 26x - 5x^2 - 3x^3}$$

To **factorise** an expression fully, it is necessary to find terms (numbers, letters or a combination) which are factors of every term in the bracket.

Eg: Fully factorise $28x^3y^2 - 12x^5y^7$

$$28x^3y^7 - 12x^5y^2 = \mathbf{4x^3y^2\left(7y^5 - 3x^2\right)}$$

Quadratics

A **quadratic function** can be written in the form $y = ax^2 + bx + c$, and the shape of the graph produced is known as a **parabola**. It is symmetrical, and resembles either ∪ (for $a > 0$) or ∩ (for $a < 0$). If it crosses the x-axis, the equation $ax^2 + bx + c = 0$ has two distinct solutions. If it only touches at one point this will be at its maximum or minimum and the equation will have one (repeated) solution. If the graph doesn't touch the x-axis at all, the equation will have no real solutions.

To **factorise a quadratic**, it is often necessary to find more than a single term, forming two brackets.

Note: If the quadratic has only an x^2 and an x term, x will be a factor, so factorising becomes simpler.

To **factorise expressions with a single x^2 term**, write out two brackets with x at the start of each. To determine the numbers to go alongside these, find two numbers that multiply to make the constant term but add to make the x coefficient.

Eg: Solve $x^2 - 5x + 4 = 0$

$x^2 - 5x + 4 = (x \ldots)(x \ldots)$ and since -1 and -4 multiply to make 4 but add to make -5:

$$x^2 - 5x + 4 = 0 \quad \Longrightarrow \quad (x-1)(x-4) = 0 \quad \Longrightarrow \quad \boldsymbol{x = 1 \quad or \quad x = 4}$$

To **factorise expressions with a number in front of x^2,** before splitting into brackets, find two numbers that multiply to make the product of the x^2 coefficient and the constant, but add to make the x coefficient. Then split the x term into these two parts, factorise the first two terms of the expression separately, then the last two terms, and finally factorise the whole thing.

Eg: Solve: $2x^2 + x - 3 = 0$

Two numbers which multiply to make $2 \times -3 = -6$ and add to make 1: 3 and -2, so:

$$2x^2 + x - 3 = 2x^2 + 3x - 2x - 3 = x(2x+3) - 1(2x+3) = (2x+3)(x-1)$$

$$\Longrightarrow \quad (2x+3)(x-1) = 0 \quad \Longrightarrow \quad \boldsymbol{x = -\frac{3}{2} \quad or \quad x = 1}$$

Another way to solve a quadratic is **completing the square**. This will work for any quadratic, but can take longer than factorising for simpler equations. Note also that completing the square is a technique that can be used to determine a maximum or minimum point on a quadratic graph without solving the equation. The aim is to write the quadratic expression in the form $p(x+q)^2 + r$, as shown below.

Eg: Solve $x^2 + 6x + 10 = 0$ by completing the square.

Step 1: Halve the x coefficient to find the number to go with x in the squared bracket:	$(x+3)^2$ *We know that if we multiply this out we would get* $x^2 + 6x + 9$
Step 2: Subtract the square of this number from the squared bracket:	$(x+3)^2 - 9$ *By subtracting the 9 we make an expression equal to* $x^2 + 6x$
Step 3: Add the constant from the original expression, and simplify:	$(x+3)^2 - 9 + 10 = \boldsymbol{(x+3)^2 + 1}$ *This is equal to* $x^2 + 6x + 10$, *but written as a completed square*
Step 4: We can then solve the equation by rearranging (notice that now x occurs just once):	$x^2 + 6x - 5 = (x+3)^2 - 9 + 1 = (x+3)^2 - 8$ $\Rightarrow \quad (x+3)^2 - 8 = 0$ $\Rightarrow \quad (x+3)^2 = 8 \quad \Rightarrow \quad x + 3 = \pm\sqrt{8} = \pm 2\sqrt{2}$ $\Rightarrow \quad \boldsymbol{x = -3 \pm 2\sqrt{2}}$

The **quadratic formula** is the result of completing the square with the general form of a quadratic, $ax^2 + bx + c = 0$. By dividing by a, completing the square and making x the subject, we get:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: Some quadratic equations have no solutions, some have exactly one, and some have two. By considering the value of $b^2 - 4ac$ we can discriminate between these three cases. If $b^2 - 4ac$ is positive, there are two distinct solutions, if 0, only one (a 'repeated root'), and if negative, no real solutions.

Formulae

To **change the subject** of a formula, first isolate the term required, then reverse the operations applied to it.

Note: If the term appears twice, it is often necessary to collect them and factorise the expression.

Eg:
Make x the subject of the formula:

$$5y - x = \frac{2x+3}{7q^2} \quad \Rightarrow \quad 7q^2(5y - x) = 2x + 3 \quad \Rightarrow \quad 35q^2y - 7q^2x = 2x + 3$$

$$\Rightarrow \quad 35q^2y - 3 = 2x + 7q^2x \quad \Rightarrow \quad 35q^2y - 3 = x(2 + 7q^2) \quad \Rightarrow \quad \frac{35q^2y - 3}{2 + 7q^2} = x$$

Rational expressions

To **simplify, add or subtract rational expressions**, simply apply the rules of fractions. You can cancel terms if they are factors of both the numerator and denominator (that is, divide top and bottom by the same thing), and to add or subtract fractions, write with a common denominator then combine the numerators by adding or subtracting. To divide by a fraction, multiply by its reciprocal.

Eg: Simplify $\frac{x^2-100}{x^2+8x-20}$

$$\frac{x^2-100}{x^2+8x-20}=\frac{(x+10)(x-10)}{(x+10)(x-2)}=\boldsymbol{\frac{x-10}{x-2}}$$

Eg: Simplify $\frac{x+4}{2x}-\frac{x-8}{x^2-4x}$

$$\frac{x+4}{2x}-\frac{x-8}{x^2-4x}=\frac{x+4}{2x}-\frac{x-8}{x(x-4)}=\frac{(x+4)(x-4)}{2x(x-4)}-\frac{2(x-8)}{2x(x-4)}$$

$$=\frac{(x+4)(x-4)-2(x-8)}{2x(x-3)}=\frac{x^2-16-2x+16}{2x(x-3)}$$

$$=\frac{x^2-2x}{2x(x-3)}=\frac{x(x-2)}{2x(x-3)}=\boldsymbol{\frac{x-2}{2(x-3)}}$$

Functions

A **function** is a mapping which can be either **one-to-one** or **many-to-one**.

Eg: $f(x)=3x-4$ is a one-to-one function, $f(x)=x^2+5$ is a many-to-one function.

The **domain** of a function is the set of **input values** it can take.
The **range** of a function is the set of **output values** it can generate.

Eg: $f(x)=\frac{5}{x^2}$ has the domain $x\neq 0$ and the range $f(x)>0$.

A function is fully defined by both a **rule** and a **domain.**

Eg: $f(x)=\frac{2x}{(x-3)}\;\; for\ x\neq 3$.

Factor theorem

The **factor theorem**:

$$(x-a)\ is\ a\ factor\ of\ polynomial\ P(x)\quad\Leftrightarrow\quad P(a)=0\ (that\ is, a\ is\ a\ root)$$

Note: This means that if we know a factor of a polynomial, we can find the corresponding root (a root is defined as the solution to $f(x)=0$; that is, the x coordinates of points where the graph $y=f(x)$ crosses the y-axis). The factor theorem works both ways, so if we know a root already, we can determine the corresponding factor.

To **factorise** a **cubic**, first find a root by trial and error, then use the factor theorem to generate a linear factor. Finally, use **inspection** to find the remaining (quadratic) factor, and factorise this, where possible.

Eg: Factorise fully $2x^3 + 3x^2 - 16x + 15$

$$P(x) = 2x^3 - x^2 - 16x + 15$$

$$P(0) = 15 \quad \Longrightarrow \quad 0 \; is\ not\ a\ root$$
$$P(1) = 0 \quad \Longrightarrow \quad 1 \; is\ a\ root$$
$$By\ the\ factor\ theorem{:} \quad P(1) = 0 \quad \Longrightarrow \quad (x-1) \;\; is\ a\ factor$$

$$2x^3 - x^2 - 16x + 15 = (x-1)(\ldots)$$

By inspection, we need $2x^2$ in the second bracket to give the $2x^3$ term in the cubic.
We also need -15 in the second bracket to give the 15 term in the cubic.

$$P(x) = (x-1)(2x^2 + \cdots - 15)$$

To get the x^2 term we need to examine the x terms from each, as well as the combination of the x^2 and constant terms. Since we already have $-1 \times 2x^2$ giving $-2x^2$, we need x^2 to give the result $-x^2$. This must be achieved by x multiplied by the second bracket's x term which therefore must be simply x.

$$P(x) = (x-1)(2x^2 + x - 15)$$

Finally, factorise the quadratic if possible:

$$P(x) = (x-1)(2x-5)(x+3)$$

Sketching curves

When **sketching a quadratic**, include:
The correct shape: A positive quadratic (one with a positive x^2 coefficient) will have a single turning point – a minimum – and a negative quadratic will have a maximum point. The graph will be symmetrical about its vertex (the max/min point), and should be a smooth curve with no sharp turns which gets steeper as it increases but never vertical.
The y-intercept: The point at which the curve crosses the y-axis. There will always be one, and only one, for a quadratic, and it is easy to identify because the curve crosses the y-axis when $x = 0$. Substituting $x = 0$ into $y = ax^2 + bx + c$ gives $y = c$, so the intercept is $(0, c)$.
Any x-axis crossing points: These are points where the curve $y = ax^2 + bx + c$ crosses the line $y = 0$, so they are solutions to $ax^2 + bx + c = 0$ and can be found – if any exist – by either factorising, completing the square or applying the quadratic formula. There may be 0, 1 or 2 roots.
[Sometimes required] **The position of the vertex:** By completing the square, it is possible to determine the coordinates of the maximum or minimum point. This allows a more precise sketch.

When **sketching a cubic**, include:
The correct shape: A positive cubic decreases without limit as x decreases, and increases without limit as x increases (so it tends to go from bottom left to top right). A negative does the opposite. Either can have a point of inflection in the middle (a 'wiggle'), like $y = x^3$, or a maximum point and a minimum point.
The y-intercept: Like with the quadratic, since this is simply the value of the cubic expression when $x = 0$, it is just the constant term.
Any x-axis crossing points: These are not always straightforward to find, and a cubic may have one, two (in this case one is also a maximum or a minimum) or three. Use factor theorem to determine the roots.
[Sometimes required] **The position of any stationary points:** If there are stationary points (some cubics have no points where the gradient is zero, some have one, some two), these can be found using differentiation and solving $\frac{dy}{dx} = 0$. Their nature (maximim, minimum or point of inflection) can be determined by considering the height (y) or gradient ($\frac{dy}{dx}$) on either side.

Simultaneous equations

One method for solving **simultaneous linear equations** is the **elimination method**. Coefficients of one of the variables are made equal and then the equations are effectively added or subtracted from one another to eliminate this variable.

Eg:

(1)	$5x - 2y = 6$
(2)	$2x + 4y = 9$
(1)×2	$10x - 4y = 12$
(2)+(1')	$12x = 21 \quad \Rightarrow \quad x = \frac{21}{12} = \frac{7}{4} = 1.75$
Sub into (2)	$2\left(\frac{7}{4}\right) + 4y = 9 \quad \Rightarrow \quad 4y = 9 - \frac{7}{2} = \frac{11}{2} \quad \Rightarrow \quad y = \frac{11}{8} = 1.375$

More generally, a wider range of **simultaneous equations** can be solved using **substitution**.

Eg:

(1)	$y - 1 = 2x$
(2)	$y^2 - 3x^2 = 6$
Rearrange (1)	$y = 2x - 1$
Sub into (2)	$(2x - 1)^2 - 3x^2 = 6$
Rearrange (2')	$x^2 - 4x - 5 = 0$
Solve (2')	$(x + 1)(x - 5) = 0 \quad \Rightarrow \quad x = -1 \quad or \quad x = 5$
Sub into (1')	$y = 2(-1) - 1 = -3 \quad or \quad y = 2(5) - 1 = 9$
Write out solutions	$x = -1 \;\; y = -3 \quad or \quad x = 5 \;\; y = 9$

Note: Take care to pair up the correct values of x and y – they represent the coordinates of the crossing points for the two equations.

Inequalities

When solving a **linear inequality**, if you **multiply or divide by a negative**, you **reverse the sign**.

Note: The reason for this is clear when you consider $3 < 5$. By subtracting 8 from each side we get $-5 < -3$, which is of course perfectly correct. But notice that this is equivalent to $-3 > -5$ which not only has different signs to the original statement but also has the inequality sign reversed.

To solve a **quadratic inequality**, it is necessary first to find the **critical values** (that is, the solutions to the related equation), then use a sketch to interpret these values as solution regions.

Eg:
Solve $x^2 - 5x + 6 \geq 0$

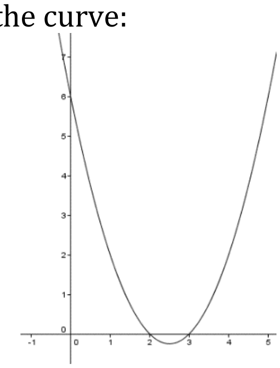
Finding critical values: $x^2 - 5x + 6 = 0$

$$\Rightarrow \quad (x-2)(x-3) = 0 \quad \Rightarrow \quad x = 2 \;\; and \;\; x = 3$$

Interpreting the sketch:

$$x \leq 2 \quad or \quad x \geq 3$$

Sketching the curve:



Indices

The **multiplication** rule:

$$a^m \times a^n = a^{m+n}$$

The **division** rule:

$$a^m \div a^n = a^{m-n}$$

The **power** rule:

$$(a^m)^n = a^{mn}$$

The **negative index** rule:

$$a^{-n} = \frac{1}{a^n}$$

The **zero index** result:

$$a^0 = 1$$

The **root** rule:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad and, more\ generally: \qquad a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Eg:

Simplify: $\frac{\sqrt[4]{x^2y^3}}{x^{a-3}}$

$$\frac{\sqrt[4]{x^2y^3}}{x^3} = x^{\frac{2}{4}}y^{\frac{3}{4}}x^{-3} = x^{\frac{1}{2}-3}y^{\frac{3}{4}} = \boldsymbol{x^{-\frac{5}{2}}y^{\frac{3}{4}}}$$

Eg 2:

Solve: $\frac{3^{3x}}{27} = \frac{1}{9^{2x+1}}$

$$\frac{3^{3x}}{27} = \frac{1}{9^{2x+1}} \quad \Rightarrow \quad \frac{3^{3x}}{3^3} = 9^{-2x-1}$$

$$\Rightarrow \quad 3^{3x-3} = (3^2)^{-2x-1} \quad \Rightarrow \quad 3^{3x-3} = 3^{-4x-2}$$

$$\Rightarrow \quad 3x - 3 = -4x - 2 \quad \Rightarrow \quad 7x = 1 \quad \Rightarrow \quad x = \frac{1}{7}$$

Algebraic proof

To prove a result, it is necessary to formulate your **assumptions** and define your **variables**. This is usually in the form of an algebraic statement such as an equation. Each line of your proof should then be the next line of a reasoned argument, with each statement being a necessary consequence of the previous one, finishing with a **conclusion**.

Eg:

Prove that the difference between two consecutive cube numbers is not a multiple of 3.

The difference between two consecutive cube numbers can be written as $(n+1)^3 - n^3$

$$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1 = 3n(n+1) + 1$$

Since $3n(n+1)$ has 3 as a factor, $3n(n+1)$ is necessarily a mutliple of 3.
Therefore $3n(n+1) + 1$ is not a multiple of 3.

Sequences

To find the n^{th} term of a **linear sequence**, identify the common difference (the difference between any consecutive terms), and compare the sequence to the related multiplication table. Add or subtract the required value to transform this sequence into yours.

Eg:
Prove that 340 is not in the linear sequence 6, 10, 14, 18, ….

$$T(n) = 4n \quad \Rightarrow \quad 4, 8, 12, \ldots \quad \Rightarrow \quad \textit{This sequence must be } T(n) = 4n + 2$$

$$4n + 2 = 340 \quad \Rightarrow \quad n = \frac{338}{4} = 84.5. \ \textit{Not a whole number} \ \Rightarrow \ 340 \ \textit{not in sequence}$$

To find the n^{th} term of a **quadratic sequence**, find the sequence of differences, then the sequence of second differences. Halve the second difference to find the n^2 coefficient. Find the difference between this simple quadratic sequence and your original sequence. Find the n^{th} term of this (linear) sequence and add it to the n^2 part.

Eg:
Find the n^{th} term rule of the quadratic sequence $12, 32, 62, 102, \ldots$.

Sequence of differences: $20, 30, 40, \ldots$
Sequence of second differences: $10, 10, 10, \ldots$
Quadratic part: $T(n) = 5n^2 \quad \Longrightarrow \quad 5, 20, 45, 80, \ldots$
Difference: $7, 12, 17, 22, \ldots \quad \Longrightarrow \quad T(n) = 5n + 2$
Complete sequence: $\boldsymbol{T(n) = 5n^2 + 5n + 2}$

Some sequences tend towards a **limit** (that is, sucessive terms get increasingly close to, but never surpass, a particular value). Depending on the n^{th} term rule for these sequences, the limit can be found through algebraic manipulation, using the fact that $\frac{1}{n}$ tends to 0 as n 'tends to ∞' (aka 'increases without limit').

Eg:
Find the limit of the sequence defined by $T(n) = \frac{8n-4}{4n+20}$

Step 1: Divide through (top and bottom of the fraction) by n:

$$T(n) = \frac{8 - \frac{4}{n}}{4 + \frac{20}{n}}$$

Step 2: Use the fact that $\frac{1}{n} \to 0$ as $n \to \infty$:

$$\frac{8 - \frac{4}{n}}{4 + \frac{20}{n}} \to \frac{8-0}{4+0} = 2 \ \ as\ n \to \infty \quad therefore\ \textbf{the limit of the sequence is 2}$$

Section 3 – Coordinate Geometry

Straight lines

The **distance between two points** can be calculated by constructing a right-angled triangle between the coordinates and applying Pythagoras' Theorem:

$$Distance\ between\ (x_1, y_1)\ and\ (x_2, y_2):\quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **midpoint** of the line between the points (x_1, y_1) and (x_2, y_2) is given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Note: This is simply the average of the x coordinates and the average of the y coordinates.

If you are required to find a point which is a given **proportion** of the way between two points, it can be helpful to think in terms of adding a certain **fraction** of the journey to the starting point.

Eg: The point $\frac{1}{3}$ of the way between (x_1, y_1) and (x_2, y_2) is: $\left(x_1 + \frac{1}{3}(x_2 - x_1), y_1 + \frac{1}{3}(y_2 - y_1)\right)$

Note: This can be simplified, in this case, to $\left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{2}{3}y_1 + \frac{1}{3}y_2\right)$, or, alternatively, the midpoint formula can be rewritten to say: $\left(x_1 + \frac{1}{2}(x_2 - x_1), y_1 + \frac{1}{2}(y_2 - y_1)\right)$.

The **gradient** of the line between the points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note: This is often described as 'y step over x step', or 'rise over run'.

Lines with gradients m_1 and m_2 are **parallel** if $m_1 = m_2$. They are **perpendicular** if $m_1 m_2 = -1$.

Note: The concepts behind the above results are more important (and more easily memorable) than the formulae used to describe them mathematically. The result below is the only one where memorising the formula gives a definite advantage to quickly solving problems, especially since simplification into a specific form is not always required.

Given the **gradient**, m, and a **single point**, (x_1, y_1), the **equation** of a line can be generated using:

$$y - y_1 = m(x - x_1)$$

Given **two points**, the **equation** of a line can be generated using:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Note: The above result is simply a combination of the definition of gradient and the gradient & point formula above. As such it is not necessary to memorise this form if you are already confident with finding the gradient between two points and can recall $y - y_1 = m(x - x_1)$.

The **crossing point** of two lines directly corresponds with the **values of x and y** which satisfy both equations. This can be found either by reading off the graph, or – more precisely – by solving the equations simultaneously.

Circles

The **equation of a circle** with centre (0,0) and radius r is given by:

$$x^2 + y^2 = r^2$$

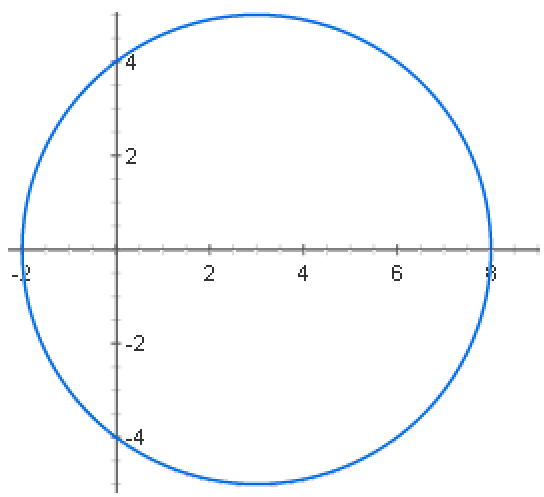
Note: This can be understood by considering any point on the circle and constructing a right-angled triangle, with the line from the origin to the point as the hypotenuse. By Pythagoras' Theorem, the square of the radius must be equal to the sum of the squares of the x and y coordinates.

The **general equation of a circle** with centre (a, b) and radius r is given by:

$$(x - a)^2 + (y - b)^2 = r^2$$

Note: This is simply a translation from the original circle – centre (0,0) – by vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Eg: The circle $(x - 3)^2 + y^2 = 25$:



Note: The most common mistakes to watch out for when interpreting a circle equation are getting the signs wrong for the centre coordinates or forgetting to square root the number on the right to get the radius.

To **sketch** a **circle**:
Step 1: Find the radius and centre from the equation.
Step 2: Mark the centre.
Step 3: Use the radius to mark on the four points directly above, below and to either side.
Step 4: Draw the circle through these four points, indicating where the circle crosses the axes, if applicable.

Note: you may also have to rely on other circle facts such as how to calculate the circumference or area, or the circle theorems (see section 6 for details).

To **find the equation of a circle** given the **end points of the diameter**, calculate the midpoint and the distance between the points. This will give you the centre and the diameter. Halve the diameter to get the radius, then put into the form $(x-a)^2+(y-b)^2=r^2$.

Like any other curve, the **crossing points** of a **line** and a **circle** can be found by solving simultaneously (using the substitution method).

A **tangent** to the circle at a particular point is a line which touches the circle only at that point. It is always **perpendicular** to the radius.

A **normal** to the circle at a particular point is a line passing through a circle which is **perpendicular** to the tangent at that point.

Note: The normal line at any point on a circle will pass through the centre, since it is perpendicular to the tangent which is perpendicular to the radius.

To find the **equation of a tangent or normal** at a particular point:
Step 1: Find the centre of the circle.
Step 2: Calculate the gradient of the line segment from the centre to your point.
Step 3i: For a normal, use this gradient and your point in the formula $y-y_1=m(x-x_1)$.
Step 3ii: For a tangent, first find the tangent gradient by using $m_1=-\frac{1}{m_2}$, then use the formula.

Note: You may need to complete the square to convert a circle equation into the preferred form first.

Eg: Find the equation of the tangent to the circle $x^2+2x+y^2-6y=25$ at the point (2,7).

$$x^2+2x+y^2-6y=25 \quad\Rightarrow\quad (x+1)^2-1+(y-3)^2-9=25 \quad\Rightarrow\quad (x+1)^2+(y-3)^2=35$$

$$Centre\text{: } (-1,3) \quad\Rightarrow\quad Grad\ of\ radius\ line\text{: } \frac{7-3}{2-(-1)}=\frac{4}{3} \quad\Rightarrow\quad Grad\ of\ tangent\text{: } -\frac{1}{\frac{4}{3}}=-\frac{3}{4}$$

$$Equation\ of\ tangent\text{: } y-7=-\frac{3}{4}(x-2) \quad\Rightarrow\quad y=-\frac{3}{4}x+\frac{17}{2}$$

Section 4 – Calculus

Differentiation is a method for finding the gradient of a curve at any given point. It can be thought of as the *rate of change* of y with respect to x.

For a **small change** in x and a corresponding small change in y, the **gradient of a chord** can be written as:

$$\frac{\delta y}{\delta x}$$

Note: Here, 'chord' refers to a straight line segment drawn between two nearby points on a curve.

Gradient

We define the **gradient** of a **curve** at a particular point as the gradient of the **tangent** to the curve at that point.

Note: As the end-points get closer together, the gradient of the chord approaches the gradient of the tangent at each point (ie, the gradient of the curve). This is the basis for differentiation – the limit to which the gradients of the ever-decreasing chords tends is the gradient of the curve at that point.

The limit of $\frac{\delta y}{\delta x}$ as $\delta y, \delta x \to 0$ is written as $\frac{dy}{dx}$ and is known as the **differential** of y with respect to x.

Note: The proof of this idea involves the concept of shrinking a small quantity until it is essentially of zero size. This is effectively finding the gradient of a chord connecting a point to *itself*. While this idea is worth being aware of, it is not necessary to memorise a proof of it.

$$y = x^n \quad \Longrightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

Note: This can be thought of as "bring the power down in front, then reduce the power by one".
Note: If there is a number multiplied by the x^n term, this is not affected by differentiating.

Eg:

$$y = 5x^4 \quad \Longrightarrow \quad \frac{dy}{dx} = 20x^3$$

$$y = f(x) \pm g(x) \quad \Longrightarrow \quad \frac{dy}{dx} = f'(x) \pm g'(x)$$

Eg: The differential of $2x^4 - 8x^2 + 4$ is $8x^3 - 16x$ (note that any constants differentiate to 0).

Note: $f'(x)$ is sometimes used to denote the derivative of $f(x)$. It is equivalent to $\frac{d(f(x))}{dx}$.

To find the gradient of a curve at a particular point, calculate $\frac{dy}{dx}$ and substitute in the x coordinate.

Eg: Find the gradient of the curve $y = (2x + 3)(x^2 - 5)$ at the point $(2, -7)$.

Multiply out:

$$y = (2x + 3)(x^2 - 5) = 2x^3 + 3x^2 - 10x - 15$$

Differentiate:

$$\frac{dy}{dx} = 6x^2 + 6x - 10$$

Substitute in $x = 2$:

$$At\ x = 2 \quad \frac{dy}{dx} = 6(2^2) + 6(2) - 10 = 26 \quad \Longrightarrow \quad Gradient = 26$$

To find a **point on a curve** with a **given gradient**, differentiate then set your expression equal to the gradient and solve for x. Finally, substitute into the original equation for a corresponding value for y.

Eg: Find any points on the curve $y = x^3 - 3x$ where the gradient is 45.

Differentiate:

$$\frac{dy}{dx} = 3x^2 - 3$$

Rewrite using $\frac{dy}{dx} = 45$:

$$45 = 3x^2 - 3 \quad \Longrightarrow \quad x^2 - 1 = 15 \quad \Longrightarrow \quad x^2 = 16 \quad \Longrightarrow \quad x = \pm 4$$

Substitute back into the original equation:

$$For\ x = 4: \quad y = 4^3 - 3(4) = 52 \quad \Longrightarrow \quad (4,52)$$

$$For\ x = -4: \quad y = (-4)^3 - 3(-4) = -52 \quad \Longrightarrow \quad (-4, -52)$$

Given the gradient of a curve at a particular point, we can find the **equation of the tangent** using $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the point on the curve and m is the gradient.

Eg: Find the equation of the tangent to the curve $y = 5x^3 - 6$ at the point $(1, -1)$.

$$\frac{dy}{dx} = 15x^2 \qquad At\ x = 1: \quad m = 15(1^2) = 15 \qquad \Longrightarrow \quad using\ y - y_1 = m(x - x_1)$$

$$y - (-1) = 15(x - 1) \quad \Longrightarrow \quad y = 15x - 16$$

Stationary points

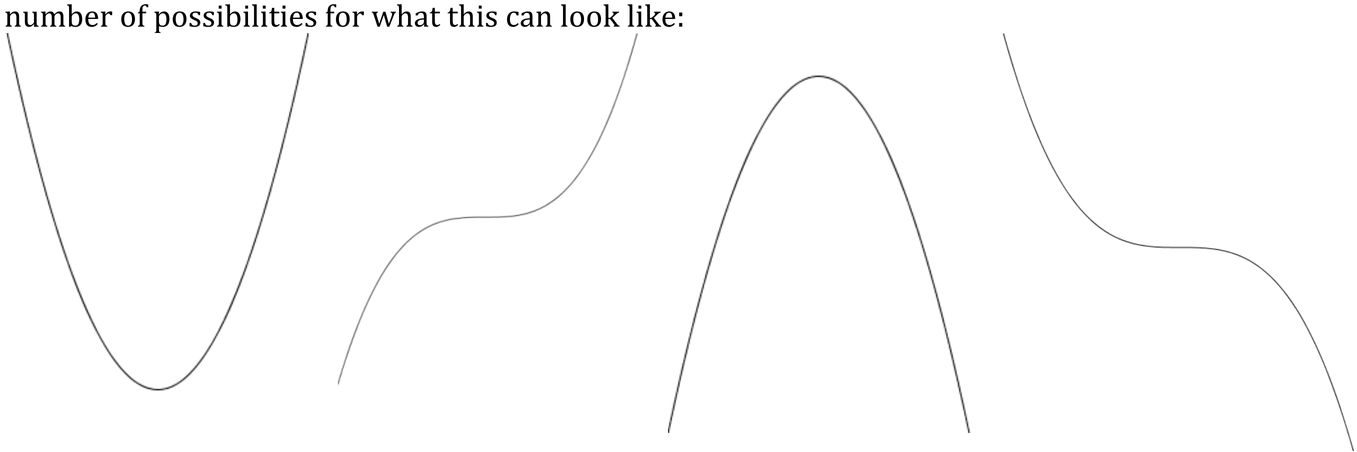
Recall that $\frac{dy}{dx}$ means the **rate of change** of y with respect to x.

$$\frac{dy}{dx} > 0 \quad \Longrightarrow \quad y \text{ increases as } x \text{ increases}$$

$$\frac{dy}{dx} < 0 \quad \Longrightarrow \quad y \text{ decreases as } x \text{ increases}$$

Note: Often the rate of change will be with respect to time.

A **stationary point** is defined to be a point on the curve where the **gradient is zero**. There are a number of possibilities for what this can look like:



Minimum   Point of inflection   Maximum   Point of inflection

To find stationary points, find $\frac{dy}{dx}$ and solve $\frac{dy}{dx} = 0$. The values of x will be the x coordinates corresponding to the stationary points.

To determine the nature of a stationary point, you can examine the gradient ($\frac{dy}{dx}$) or the height (y) of the function just before and just after the stationary point. For a **maximum**, the gradient changes from negative to positive, for a **minimum**, from positive to negative, and for a **point of inflection** the gradient either goes from positive to zero and back to positive or from negative to zero and back to negative.

Note: When sketching a graph with known stationary points, mark on their coordinates, determine their nature, and include any x-axis and y-axis crossing points.

Section 5 – Matrix Transformations

Terminology

A **matrix** is a rectangular array of numbers. Each entry in the matrix is called an **element**.

A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.

We can add or subtract matrices provided they have the **same order**.

To add or subtract matrices, all that is required is that we add or subtract **corresponding elements** from each matrix.

Eg:

$$\begin{bmatrix} 3 & 2 & 5 \\ -1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 1 \\ 0 & -4 & 10 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 4 \\ -1 & 4 & -8 \end{bmatrix}$$

Multiplication

To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.

Note: This is comparable to finding the scalar multiple of a vector – in fact, that process is just a specific example of multiplying a matrix by a constant.

Eg:

$$5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 5 \end{bmatrix}$$

We can multiply two matrices **A** and **B** only if the number of columns of **A** equals the number of rows of **B**.

Note: This does not mean the matrices have to be of equal order, but due to the method used for multiplying matrices, they must fulfil this requirement. You will only be required to multiply a 2 by 2 matrix by either a 2 by 1 vector or another 2 by 2 matrix.

To multiply two square matrices of the same order, calculate the value of the each element by finding the sum of the products of the elements in the corresponding row of the first matrix and the corresponding column of the second.

That is, to find the element in the first row and second column, you would multiply each element in the first matrix's first row by the corresponding element in the second matrix's second column.

Eg:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \quad \boldsymbol{AB} = \begin{bmatrix} 2 \times -1 + 0 \times 3 & 2 \times 0 + 0 \times 2 \\ -1 \times -1 + 1 \times 3 & 1 \times 0 + 1 \times 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & 2 \end{bmatrix}$$

In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. Matrix multiplication is not, in general, commutative.

Note: Despite this, you will find that, in the limited case of *some* of the transformation matrices you will see, the order does not make a difference.

There is a square matrix I with the property that $IA = AI = A$ for any compatible matrix A (that is, square, and with the same order as I). This matrix is known as the **identity matrix**.
The 2 *by* 2 identity matrix is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ **Identity matrix.**
Right remains right, up remains up.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ Reflection in the y-axis.
Right has become left, up remains up.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Reflection in the x-axis.
Right remains right, up has become down.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Rotation by $180°$
Right has become left, up has become down.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Reflection in the line $y = x$.
Right has become up, up has become right.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Rotation by 90° anticlockwise.
Right has become up, up has become left.

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Rotation by 90° clockwise.
Right has become down, up has become right.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Reflection in the line $y = -x$.
Right has become down, up has become left.

$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ Enlargement by scale factor a in the x direction.
Right is multiplied by a, up remains up.

$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ Enlargement by scale factor a in the y direction.
Right remains right, up is multiplied by a.

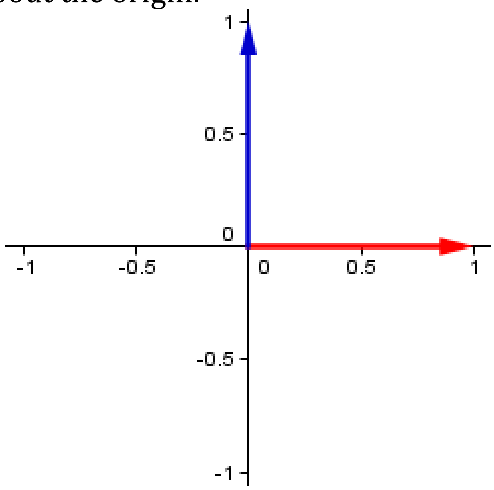
$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ Enlargement by scale factor a from the origin.
Right is multiplied by a, up is multiplied by a.

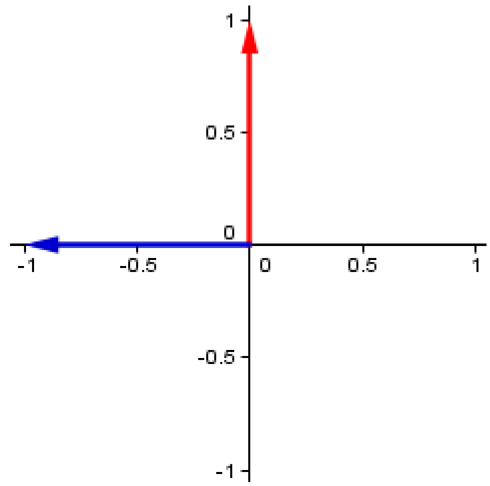
$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ Enlargement by scale factor a in the x direction and scale factor b in the y direction.
Right is multiplied by a, up is multiplied by b.

Note: The matrices shown above all either rotate, reflect or enlarge with respect to the origin (or in lines through the origin). You need to be able to identify them and to generate them for yourself.

Each column of a 2 *by* 2 transformation matrix is the **image** of the corresponding column of I (the identity matrix).

Eg:
If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the rightwards direction has been converted to upwards.
And we have $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, meaning 'up' has been converted to 'left'. This describes a rotation of 90° clockwise about the origin.





The image on the left shows the original vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The image on the right shows the transformed vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

When combining two transformations in matrix form, their effect can be combined into one single matrix by matrix multiplication. The order is important, so remember that the latest matrix to be applied is always applied on the left.

Eg: Find a matrix that describes a stretch by scale factor 3 in the y-direction from the origin followed by a rotation by 90° clockwise about the origin.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$$

Section 6 –Geometry

In addition to basic formulae for rectangles and triangles you should recall that the area of a triangle can be found using:

$$Area = \frac{1}{2}ab\sin C$$

Where a and b are two lengths of the triangle and C is the size of the angle in between.

For right-angled triangles, Pythagoras' theorem applies:

$$a^2 + b^2 = c^2$$

Where a and b are the lengths of the two perpendicular sides and c is the length of the hypotenuse.

For *any* triangle, whose sides are of length a, b and c with opposite angles A, B and C respectively:

$$Sine\ rule: \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$Cosine\ rule: \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Note: The sine and cosine rule (as well as the general formula for area) are provided on the formula sheet (see the end of this booklet). You will need to be confident determining which to use and how.

To find **area**, first see if you can use $Area = \frac{1}{2}bh$. If not, try $Area = \frac{1}{2}ab\sin C$.

To find a **length**, first see if you can use *Pythagoras* (triangle must be right-angled, and you will need to know two of the sides). If not, try *Sine rule* (you will need an opposite side and angle pair, plus the angle opposite your desired length). If not, try *Cosine rule* (you will need the other two sides and the angle in between them).

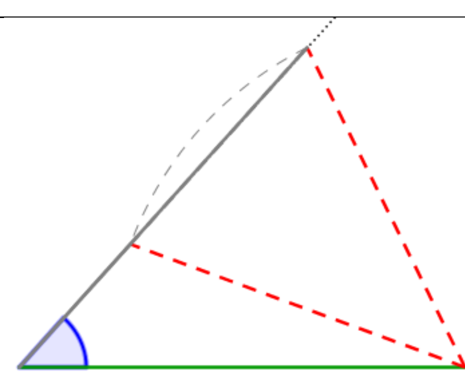
To find an **angle**, first see if you can use the *Angle sum* of triangles. If not, try *Sine rule* (you will need an opposite side and angle pair, plus the side opposite your desired angle). If not, try *Cosine rule* (you will need all three sides).

Note: Sometimes you may need to apply a number of these rules to find what you want. If in doubt, use the simplest first and work out any unknown values you can, then reevaluate.

The **ambiguous case** can occur, usually when applying the sine rule, when calculating an angle.

If we know the length of one side and the angle opposite, and one other side, we can find the *sine* of the angle opposite, but this may yield two equally valid possible solutions.

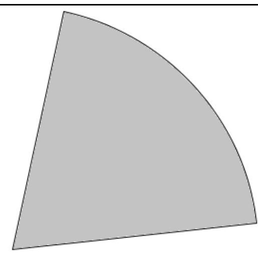
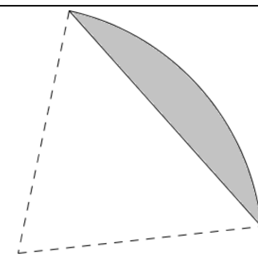
Recall that $\sin 80 = \sin 100$, etc, so for every acute angle that the calculator will give, there is an alternative obtuse angle which can be found using $90 - \theta$.



Pythagoras' theorem, and right-angled trigonometry may need to be applied in **3 dimensions**:

- For the angle between **a line and a line**: construct a triangle within the 3-D shape, and apply the normal rules.
- For the angle between **a line and a plane**: drop a perpendicular line down to the plane from the line and form a right-angled triangle. The angle you want is between the side of the triangle which lies on the plane and your original line.
- For the angle between **a plane and a plane:** find a line in each plane which is perpendicular to the line of intersection of the two planes. The angle between these lines is the angle between the planes.

The circumference of a circle is given by $C = 2\pi r$ and the area by $A = \pi r^2$. These formulae can be applied to **sectors** and **segments** as shown below:

	Sector	Segment
		
Area:	$A = \frac{\theta}{360}\pi r^2$	$A = \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2 \sin\theta$
Perimeter:	$P = \frac{\theta}{360}2\pi r$	$P = \frac{\theta}{360}2\pi r + r\sqrt{2 - 2\cos\theta}$

Note: The more complicated-looking formulae for the segment do **not** need to be learned or even applied in this form – they are a combination of the area of a triangle formula and the cosine rule.

Know and be able to apply the following **circle facts**:

- A triangle with two corners on the circumference and one in the centre of a circle is isosceles.
- A tangent and a radius always meet at right angles.
- The triangle formed by two tangents and the chord in between is isosceles.
- If a radius bisects a chord it does so at right angles, and vice versa.

Know, apply and quote the following **circle theorems**:

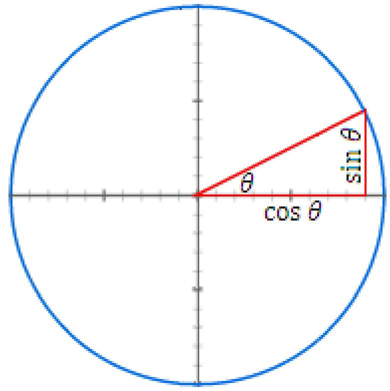
- The angle at the centre of a circle is twice the angle at the circumference.
- The angle in a semicircle is a right angle.
- Angles in the same segment are equal.
- The sum of the opposite angles of a cyclic quadrilateral is 180°.
- The angle between a chord and the tangent is equal to the angle in the alternate segment.

Note: This final theorem can be quoted by name. Eg "By the Alternate Segment Theorem, $x = 30°$.

The *sine* function can be thought of as the height above the centre of a point moving around a circle. The *cosine* function can be thought of as the distance to the right of the centre.

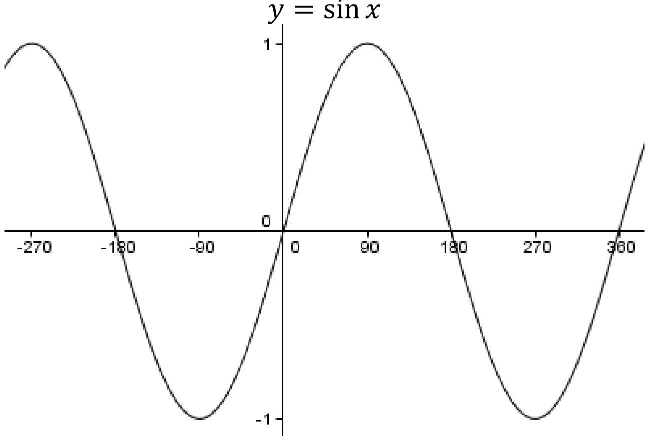
The angle in question is the anti-clockwise angle from the x axis as shown.

Note that the height (the *sine* function) will be negative between 180° and 360°, and the distance to the right (the *cosine* function) will be negative between 90° and 270°.

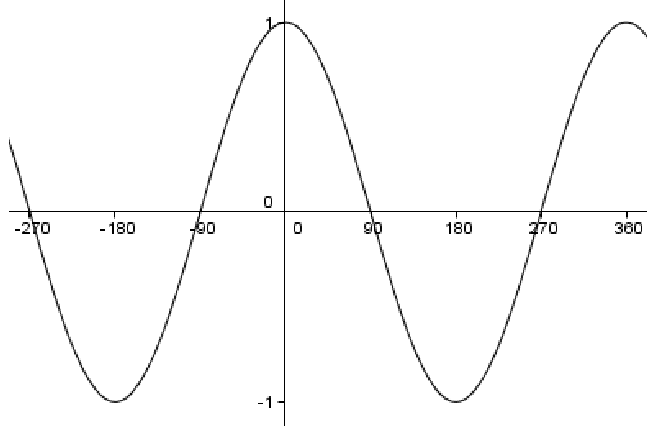


It is important to note that the functions $\sin x$, $\cos x$ and $\tan x$ are valid not just for angles from 0° to 90°. Using the idea of the circle above, we can generate graphs for each function. Also, since it is possible to go as far clockwise or anticlockwise around the circle as you like, the function, and therefore graph, extends infinitely in both directions, but will simply repeat the first 360°.

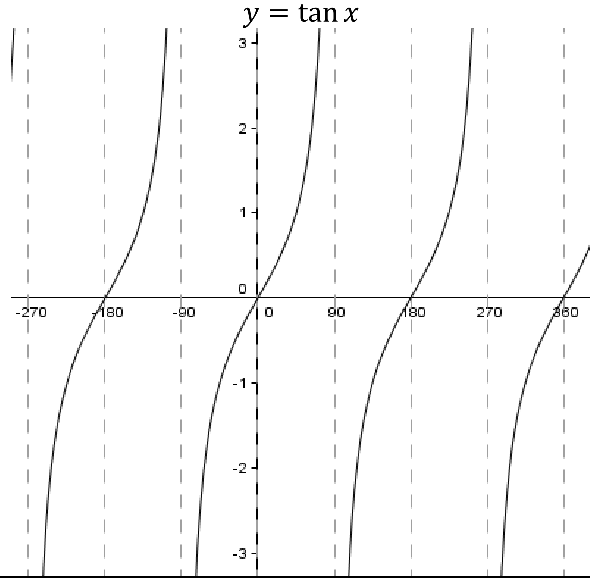
$y = \sin x$



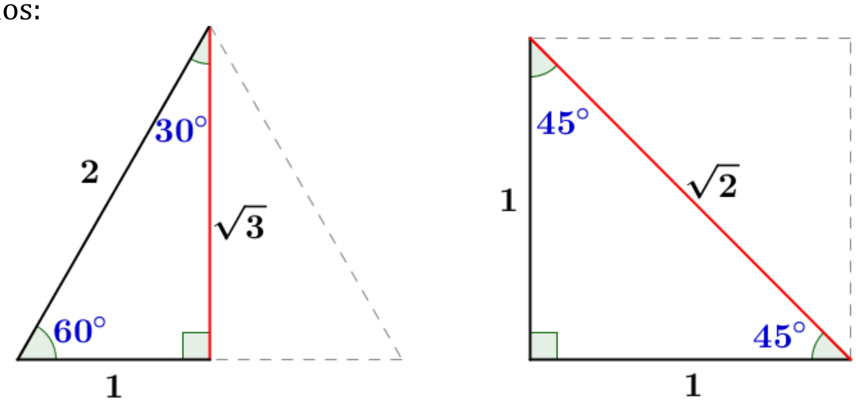
$y = \cos x$



$y = \tan x$



While approximate values for $\sin x$ can be found using a calculator, sometimes exact values are useful. By cutting either an equilateral triangle or a square in half, we can use Pythagoras' theorem to determine the exact lengths of the sides formed, and hence the exact values of a number of trigonometric ratios:



Angle	Sine	Cosine	Tangent
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	$(undefined)$

For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note: This can be seen from the right-angled trigonometry formulae, since $\frac{\left(\frac{opp}{hyp}\right)}{\left(\frac{adj}{hyp}\right)} = \frac{opp}{adj}$

For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1$$

Note: This can be seen by applying Pythagoras' theorem to a standard right-angled triangle:

$$\left(\frac{opp}{hyp}\right)^2 + \left(\frac{adj}{hyp}\right)^2 = \frac{opp^2 + adj^2}{hyp^2} = \frac{hyp^2}{hyp^2} = 1$$

Note: The two trigonometric identities above will be provided in the formula sheet.

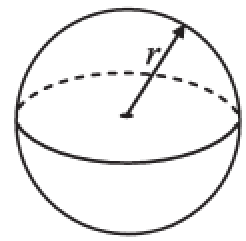
To solve a trigonometric equation, first rearrange into the form $\sin x = k$ (eg $\sin x = 0.5$). Then find the principal value using a calculator (ie $x = \sin^{-1} 0.5 = 30°$), then use the graph to find any other valid solutions within the range required. (eg $x = 30°$ or $x = 150°$).

Appendix – Formulae Sheet

In the exam you will be provided with a sheet of formulae just as in the GCSE exam, but with a few additions. The following is a list of the results you will be given:

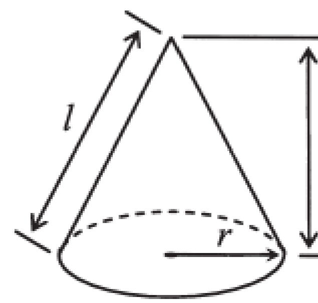
$\boldsymbol{Volume\ of\ sphere} = \frac{4}{3}\pi r^3$

$\boldsymbol{Surface\ area\ of\ sphere} = 4\pi r^2$



$\boldsymbol{Volume\ of\ cone} = \frac{1}{3}\pi r^2 h$

$\boldsymbol{Curved\ surface\ area\ of\ cone} = \pi r l$

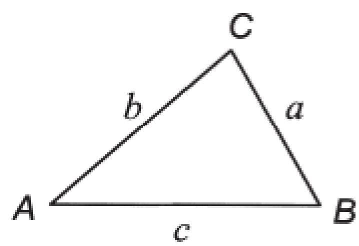


$\boldsymbol{Area\ of\ triangle} = \frac{1}{2}ab\sin C$

$\boldsymbol{Sine\ rule}$: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\boldsymbol{Cosine\ rule}$: $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$



$\boldsymbol{The\ quadratic\ equation}$: $The\ solutions\ of\ ax^2 + bx + c = 0, where\ a \neq 0, are\ given\ by$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\boldsymbol{Trigonometric\ identities}$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

Produced by A. Clohesy; TheChalkface.net
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