Further Maths Level 2 Exam Paper Solutions (Jan 2013)

Paper 1 – Non-Calculator

1 The line $y = mx + c$ passes through the point $(4, 3)$.
It is parallel to the line $y = 5x + 6$

Work out the values of m and c.

....

$m =$ ...., $c =$ .... *(3 marks)*

Since parallel to $y = 5x + 6$, the gradient must be the same, so $\boldsymbol{m = 5}$.

Substituting in the values $x = 4$ and $y = 3$:

$$y = 5x + c \quad \rightarrow \quad 3 = 5(4) + c \quad \rightarrow \quad \boldsymbol{c = -17}$$

2 The matrix $\begin{pmatrix} 5 & b \\ 4 & -1 \end{pmatrix}$ maps the point $(a, 2)$ onto the point $(28, 18)$,

such that $\begin{pmatrix} 5 & b \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 28 \\ 18 \end{pmatrix}$

Work out the values of a and b.

$a =$ ...., $b =$ .... *(4 marks)*

$$\begin{bmatrix} 5 & b \\ 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} 5a + 2b \\ 4a - 2 \end{bmatrix} \quad \Longrightarrow \quad 5a + 2b = 28 \quad and \quad 4a - 2 = 18$$

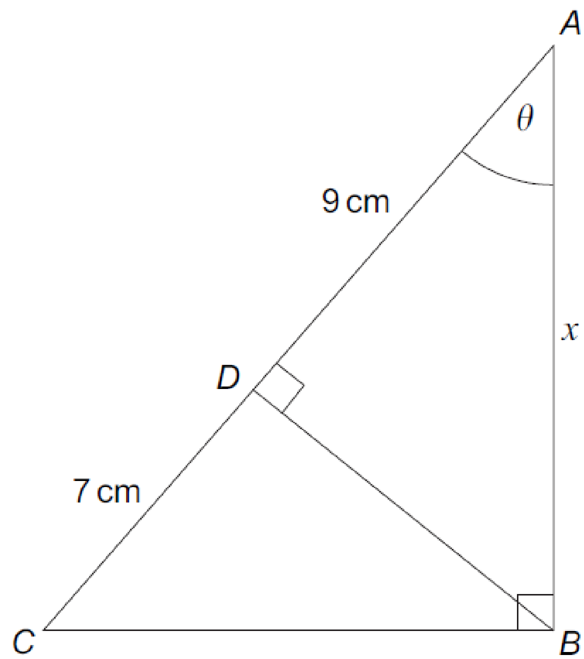
From the second equation:

$$4a - 2 = 18 \quad \Longrightarrow \quad \boldsymbol{a = 5}$$

Then, substituting into the first equation:

$$5(5) + 2b = 28 \quad \Longrightarrow \quad \boldsymbol{b = \frac{3}{2}}$$

3 *ABC* is a right-angled triangle.
D is a point on *AC*.
BD is perpendicular to *AC*.



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3 (a) Use triangle *ABC* to write $\cos\theta$ in terms of x.

....

$\cos\theta =$ .... *(1 mark)*

3 (b) By writing another expression for $\cos\theta$ in terms of x, or otherwise, work out the value of x.

....

$x =$ .... cm *(2 marks)*

a)

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{7+9} \quad \Longrightarrow \quad \boldsymbol{\cos\theta = \frac{x}{16}}$$

b)
Considering triangle ABD:

$$\cos\theta = \frac{adj}{hyp} = \frac{9}{x} \quad \Longrightarrow \quad \frac{9}{x} = \frac{x}{16} \quad \Longrightarrow \quad 144 = x^2 \quad \Longrightarrow \quad \boldsymbol{x = 12}$$

4 $w \blacktriangledown h$ is defined as $5w^2 - 8w + h^2 - 2h$

For example $1 \blacktriangledown 6 = 5 \times 1^2 - 8 \times 1 + 6^2 - 2 \times 6$

$= 5 - 8 + 36 - 12$

$= 21$

4 (a) Work out $2 \blacktriangledown 4$

....

....

....

Answer.. *(2 marks)*

4 (b) Solve $x \blacktriangledown 3 = 0$

....

Answer.. *(4 marks)*

a)

$$2 \vee 4 = 5(2)^2 - 8(2) + 4^2 - 2(4) = 20 - 16 + 16 - 8 = \mathbf{12}$$

b)

$$5x^2 - 8x + 3^2 - 2(3) = 0$$

$$5x^2 - 8x + 3 = 0$$

$$(5x - 3)(x - 1) = 0$$

$$\boldsymbol{x = \frac{3}{5} \quad or \quad x = 1}$$

5 (a) n is a positive integer.

Write down the **next** odd number after $2n - 1$

Answer .... *(1 mark)*

5 (b) Prove that the product of two consecutive odd numbers is **always** one less than a multiple of 4 .

....

(3 marks)

a)

Odd numbers form a sequence that goes up by 2 from one term to the next, so:

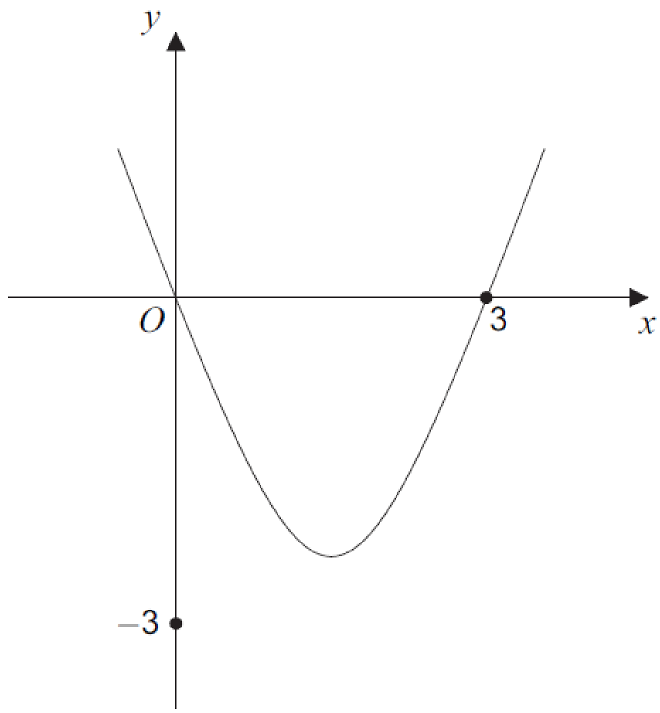
$$2n - 1 + 2 = \mathbf{2n + 1}$$

b)

$$(2n-1)(2n+1) = 4n^2 - 2n + 2n - 1 = 4n^2 - 1 = \boldsymbol{4k - 1 \; for \; some \; integer \; k}$$

Therefore the product of two consecutive odd numbers is always one less than a multiple of 4.

6 The diagram shows a sketch of $y = x^2 - 3x$



6 (a) Sketch the line $y = \frac{1}{2}(x-3)$ on the diagram.

Mark the value where this line crosses the y-axis. *(2 marks)*

6 (b) By factorising $x^2 - 3x$, or otherwise, work out the smaller solution of

$$x^2 - 3x = \frac{1}{2}(x-3)$$

....

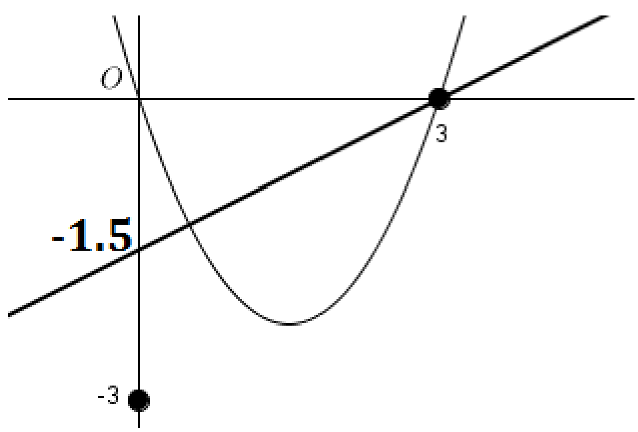
$x =$ .... *(2 marks)*

a)

Substitute in key values like $x = 0$ and $x = 3$:

$x = 0 \implies y = -\frac{3}{2} \implies \left(0, -\frac{3}{2}\right)$ *is on the line*

$x = 3 \implies y = 0 \implies (3,0)$ *is on the line*



b)

$$x^2 - 3x = \frac{1}{2}(x-3)$$

$$x(x-3) = \frac{1}{2}(x-3)$$

$$x(x-3) - \frac{1}{2}(x-3) = 0$$

$$(x-3)\left(x - \frac{1}{2}\right) = 0$$

$$x = 3 \quad or \quad \boldsymbol{x = \frac{1}{2}}$$

7 $y = \dfrac{2x^2(3x^3 - 7x)}{x}$

Work out $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} =$ .... *(4 marks)*

Simplifying:

$$y = \frac{2x^2(3x^3 - 7x)}{x} = 2x(3x^3 - 7x) = 6x^4 - 14x^2$$

Differentiating:

$$\frac{dy}{dx} = \boldsymbol{24x^3 - 28x}$$

8 $f(x)$ is a decreasing function.

$$f(x) = b - ax \quad \text{for} \quad 4 \leqslant x < 8$$

The range of $f(x)$ is $5 < f(x) \leqslant 7$

Work out the values of a and b.

....

$a =$ ...., $b =$ .... *(4 marks)*

Since $f(x)$ is decreasing, the maximum value of the range must correspond with the minimum value for the domain and vice versa.

Therefore $f(4) = 7$ and $f(8) = 5$

Substituting in these values:

$$b - 4a = 7 \quad and \quad b - 8a = 5$$

Solving simultaneously (by elimination):

$$4a = 2 \quad \Longrightarrow \quad \boldsymbol{a = \frac{1}{2}}$$

And by substitution:

$$\boldsymbol{b = 9}$$

9 Bag *A* contains $7x$ counters.

Bag *B* contains $2x$ counters.

Five counters are taken from bag *A* and put in bag *B*.

9 (a) Write an expression, in terms of x, for the number of counters now in bag *B*.

Answer.. *(1 mark)*

9 (b) The ratio of counters in bag *A* to bag *B* is now 8 : 3

Use algebra to work out the **total** number of counters in the bags.

....

Answer.. *(4 marks)*

a)

$$2x + 5$$

b)
Counters in bag A:

$$7x - 5$$

Ratio of counters in bag A to bag B:

$$\frac{7x - 5}{2x + 5} = \frac{8}{3}$$

$$3(7x - 5) = 8(2x + 5)$$

$$21x - 15 = 16x + 40$$

$$5x = 55$$

$$x = 11$$

Finally, to find the total number of counters:

$$Bag\ A\ counters + Bag\ B\ counters = 7x + 2x = 9x = \mathbf{99\ counters}$$

10 Solve the simultaneous equations

$$\frac{x-1}{y-2} = 3 \qquad \frac{x+6}{y-1} = 4$$

Do **not** use trial and improvement.
You **must** show your working.

....

$x =$ ...., $y =$ .... *(5 marks)*

Rearranging each equation into a more standard format:

$$\frac{x-1}{y-2} = 3 \quad \Longrightarrow \quad x - 1 = 3(y-2) \quad \Longrightarrow \quad x - 1 = 3y - 6 \quad \Longrightarrow \quad x - 3y = -5$$

$$\frac{x+6}{y-1} = 4 \quad \Longrightarrow \quad x + 6 = 4(y-1) \quad \Longrightarrow \quad x + 6 = 4y - 4 \quad \Longrightarrow \quad x - 4y = -10$$

Solving using elimination (subtracting the second equation from the first):

$$\boldsymbol{y = 5}$$

Substituting into first equation:

$$x - 3(5) = -5 \quad \Longrightarrow \quad \boldsymbol{x = 10}$$

11 Write $\sqrt{500} - 2\sqrt{45}$ in the form $a\sqrt{5}$ where a is an integer.

....

....

....

....

Answer.. *(2 marks)*

$$\sqrt{500} - 2\sqrt{45} = \sqrt{100 \times 5} - 2\sqrt{9 \times 5} = 10\sqrt{5} - 2(3\sqrt{5}) = 10\sqrt{5} - 6\sqrt{5} = \boldsymbol{4\sqrt{5}}$$

12 Simplify fully $\dfrac{4x^2 + 19x - 5}{9x^2 - 16} \div \dfrac{x + 5}{3x - 4}$

....

Answer.. *(5 marks)*

Factorising where possible:

$$\frac{4x^2 + 19x - 5}{9x^2 - 16} \div \frac{x + 5}{3x - 4} = \frac{(x + 5)(4x - 1)}{(3x - 4)(3x + 4)} \div \frac{x + 5}{3x - 4}$$

Eliminating the division sign:

$$\frac{(x + 5)(4x - 1)}{(3x - 4)(3x + 4)} \div \frac{x + 5}{3x - 4} = \frac{(x + 5)(4x - 1)}{(3x - 4)(3x + 4)} \times \frac{3x - 4}{x + 5}$$

Combining fractions and cancelling down:

$$\frac{(x + 5)(4x - 1)}{(3x - 4)(3x + 4)} \times \frac{3x - 4}{x + 5} = \frac{(x + 5)(4x - 1)(3x - 4)}{(3x - 4)(3x + 4)(x + 5)} = \boldsymbol{\frac{4x - 1}{3x + 4}}$$

13 $y = 2x^3 - 12x^2 + 24x - 11$

13 (a) Work out $\frac{dy}{dx}$

Give your answer in the form $\frac{dy}{dx} = a(x - b)^2$, where a and b are integers.

$\frac{dy}{dx} =$ .... *(3 marks)*

13 (b) Hence, or otherwise, work out the coordinates of the stationary point of

$$y = 2x^3 - 12x^2 + 24x - 11$$

Answer (........................, ...) *(2 marks)*

13 (c) Explain how you know that this stationary point is a point of inflection.

....

(1 mark)

a)
Differentiating:

$$y = 2x^3 - 12x^2 + 24x - 11 \quad \Longrightarrow \quad \frac{dy}{dx} = 6x^2 - 24x + 24$$

Completing the square:

$$\frac{dy}{dx} = 6x^2 - 24x + 24 = 6[x^2 - 4x + 4] = 6[(x-2)^2 - 4 + 4] = \mathbf{6(x-2)^2}$$

b)
Stationary points occur when $\frac{dy}{dx} = 0$:

$$6(x-2)^2 = 0 \quad \Longrightarrow \quad x = 2$$

Substituting into the original curve equation:

$$y = 2(2^3) - 12(2^2) + 24(2) - 11 = 16 - 48 + 48 - 11 = 5 \quad \Longrightarrow \quad \mathbf{(2, 5)}$$

c)
The curve is a cubic. If it had two stationary points, one would be a maximum, the other a minimum. Since it has just one, it must be a point of inflection. Note that a cubic may have no stationary points at all. Another explanation is that the gradient, which is $6(x-2)^2$, is never negative, so the curve must be increasing both before and after the stationary point.

14 $x^2 - 2x + y^2 - 6y = 0$ is the equation of a circle.

By writing the equation in the form $(x-a)^2 + (y-b)^2 = r^2$
work out the centre and radius of the circle.

....

Centre = $\left(\text{.......................}, \text{.......................} \right)$

Radius = ... *(5 marks)*

Completing the square:

$$x^2 - 2x + y^2 - 6y = 0$$

$$(x-1)^2 - 1 + (y-3)^2 - 9 = 0$$

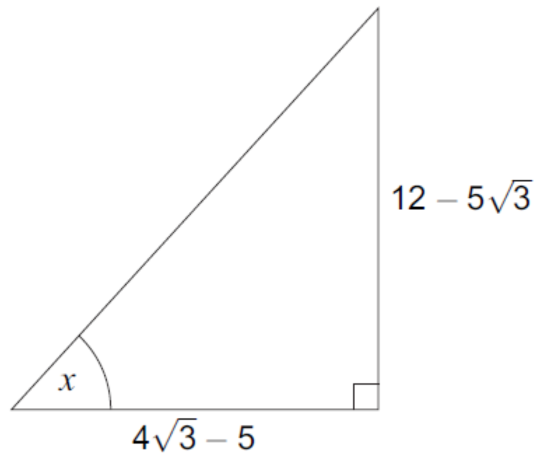
$$(x-1)^2 + (y-3)^2 = 10$$

Interpreting the standard circle equation:

$$Centre: \quad (\mathbf{1}, \mathbf{3})$$

$$Radius: \quad \sqrt{\mathbf{10}}$$

15 Show that angle $x = 60°$



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You **must** show your working.

....

(4 marks)

Using right-angled trigonometry:

$$\tan x = \frac{opp}{adj} = \frac{12 - 5\sqrt{3}}{4\sqrt{3} - 5}$$

Rationalising the denominator:

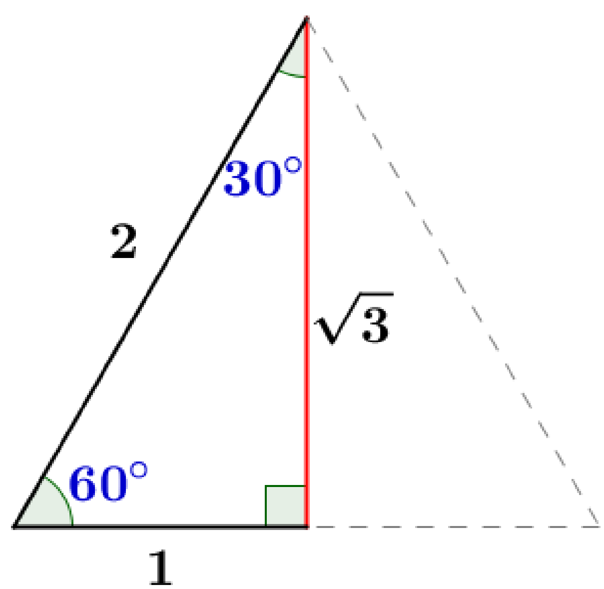
$$\frac{12 - 5\sqrt{3}}{4\sqrt{3} - 5} = \frac{(12 - 5\sqrt{3})(4\sqrt{3} + 5)}{(4\sqrt{3} - 5)(4\sqrt{3} + 5)} = \frac{48\sqrt{3} + 60 - 60 - 25\sqrt{3}}{48 - 25} = \frac{23\sqrt{3}}{23} = \sqrt{3}$$

Considering an equilateral triangle of side length 2 bisected vertically, we can see that:

$$\tan 60 = \frac{opp}{adj} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Therefore $\tan^{-1}\sqrt{3} = 60°$

Hence **angle x must be 60°**

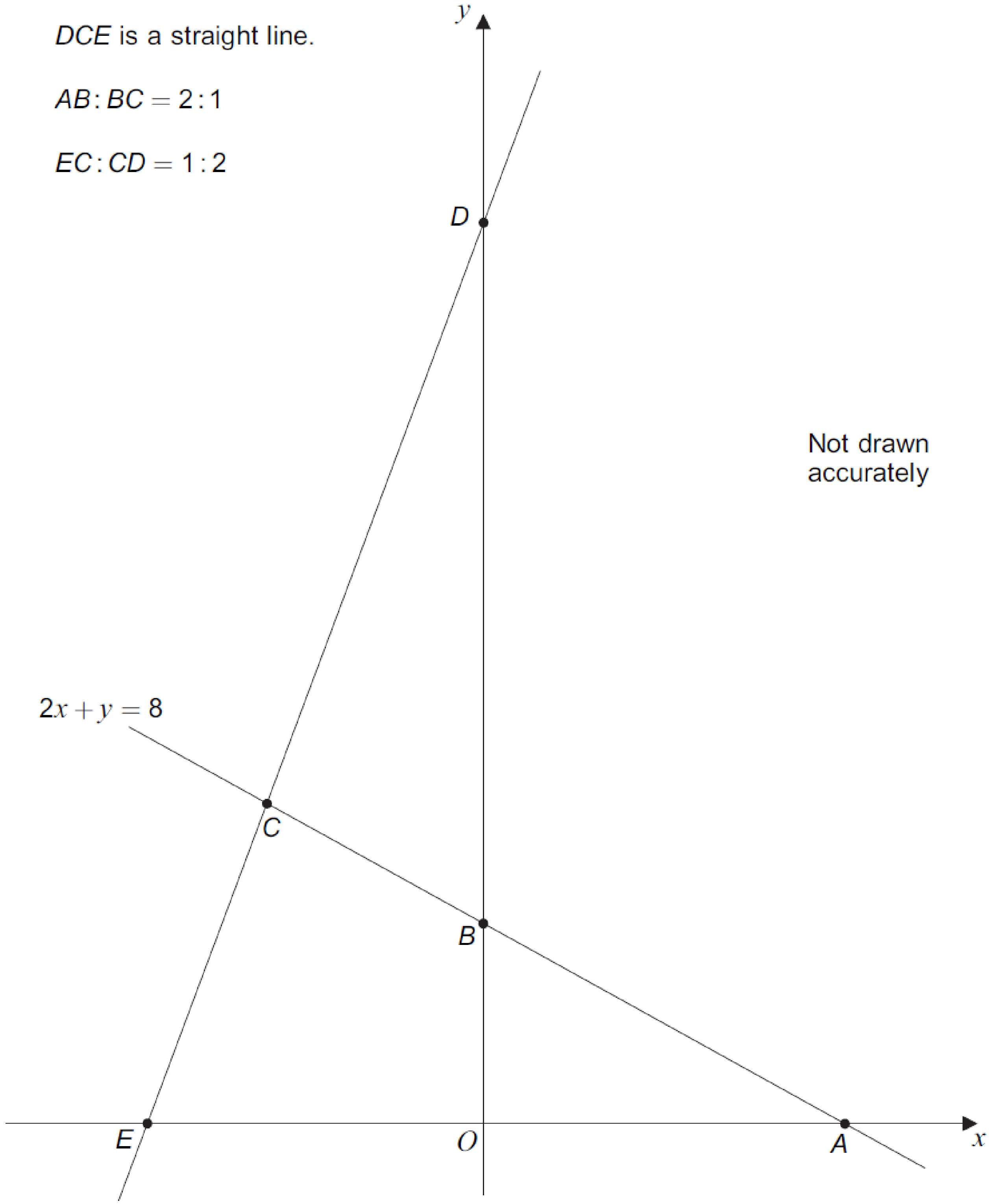


16 A, B and C are points on the line $2x + y = 8$

DCE is a straight line.

$AB : BC = 2 : 1$

$EC : CD = 1 : 2$



Work out the ratio Area of triangle AEC : Area of triangle BCD

Give your answer in its simplest form.

....

Answer .... : .... *(6 marks)*

Since A and B are on the line $2x + y = 8$, we can find their coordinates:

Point A:

$$y = 0 \quad \Longrightarrow \quad 2x + 0 = 8 \quad \Longrightarrow \quad x = 4 \quad \Longrightarrow \quad A: \quad (4,0)$$

Point B:

$$x = 0 \quad \Longrightarrow \quad 0 + y = 8 \quad \Longrightarrow \quad y = 8 \quad \Longrightarrow \quad B: \quad (0,8)$$

Now, since $AB: BC$ is $2: 1$, this must also be the ratio of their horizontal distances:
Since A is 4 to the right of B, C must be 2 to the left of B.

Similarly, we can consider their vertical distances:
Since A is 8 below B, C must be 4 above B.

This gives the position of point C:

$$C: \quad (-2,12)$$

Next, consider the point D. Since $EC: CD$ is $1: 2$, D is 3 times as further up from E as C:

$$D: \quad (0,36)$$

Also, D is twice as far to the right of C as E is to the left:

$$E: \quad (-3,0)$$

By considering the y coordinate of C (the perpendicular height) and the difference between the x coordinates of A and E (the base) we can find the area of triangle AEC:

$$Area_{AEC} = \frac{12 \times 7}{2} = 42\, units$$

By considering the magnitude of the x coordinate of C (the perpendicular height) and the difference between the y coordinaes of B and D (the base) we can find the area of triangle BCD:

$$Area_{BCD} = \frac{2 \times 28}{2} = 28\, units$$

Finally, as a ratio in its simplest form:

$$Area_{AEC}: Area_{BCD} \quad \Longleftrightarrow \quad 42: 28 \quad \Longleftrightarrow \quad \mathbf{3}: \mathbf{2}$$