Completing the Square to solve equations

Not every quadratic equations can be solved by factorising, but all can be written in *completed square form*, allowing us to find solutions even when they're not nice numbers.

Write the following expression in completed square form: $\boldsymbol{x^2 + 8x - 5}$

Step 1: Halve the x coefficient to find the number to go with x in the squared bracket:

$(x+4)^2$ *We know that if we multiply this out we would get* $x^2 + 8x + 16$.

Step 2: Subtract the square of this number from the squared bracket:

$(x+4)^2 - 16$ *By subtracting the 16 we turn the expression into* $x^2 + 8x$.

Step 3: Add on the number part from the original expression, and simplify:

$(x+4)^2 - 16 - 5 = \boldsymbol{(x+4)^2 - 21}$ *This is the* ***completed square form***.

A quadratic equation written in completed square form can be solved just by rearranging*:

$$(x+4)^2 - 21 = 0 \implies (x+4)^2 = 21 \implies x + 4 = \pm\sqrt{21} \implies \boldsymbol{x = -4 \pm \sqrt{21}}$$

1 Solve these equations by completing the square.
Give your solutions in surd form.

a $x^2 + 6x - 10 = 0$ **b** $x^2 - 4x - 7 = 0$ **c** $x^2 - 6x + 1 = 0$

d $x^2 - 10x + 5 = 0$ **e** $x^2 + 2x - 5 = 0$ **f** $x^2 - 10x + 4 = 0$

If a quadratic equation is not already in the 'standard' form above, rearrange first:

$$2x(x-1) = 3 \implies 2x^2 - 2x = 3 \implies 2x^2 - 2x - 3 = 0 \implies \boldsymbol{x^2 - x - \frac{3}{2} = 0}$$

…and even if the x coefficient is an awkward number, you can still complete the square …

$$x^2 - x - \frac{3}{2} = 0 \implies \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{3}{2} = 0 \implies \left(\boldsymbol{x - \frac{1}{2}}\right)^2 - \boldsymbol{\frac{7}{4}} = 0$$

…and rearrange to solve, as before:

$$\left(x - \frac{1}{2}\right)^2 - \frac{7}{4} = 0 \implies \left(x - \frac{1}{2}\right)^2 = \frac{7}{4} \implies x - \frac{1}{2} = \pm\frac{\sqrt{7}}{2} \implies \boldsymbol{x = \frac{1}{2} \pm \frac{\sqrt{7}}{2}}$$

2 Solve these equations by completing the square. Give your solutions correct to 3 significant figures.

a $x^2 + 6x = 12$ **b** $x^2 - 4x = 10$ **c** $x^2 - 12x = 6$

d $8x - x^2 = 11$ **e** $x^2 - 9x = 5$ **f** $x(x - 5) = 7$

Bonus info: The Quadratic Formula

The quadratic formula is what you get when you complete the square with a, b and c instead of numbers:

$$ax^2 + bx + c = 0 \implies a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You will **not** be expected to know or use this result!

1 a $-3 \pm \sqrt{19}$
b $2 \pm \sqrt{11}$
c $3 \pm \sqrt{8}$
d $5 \pm \sqrt{20}$
e $-1 \pm \sqrt{6}$
f $5 \pm \sqrt{21}$

2 a 1.58, −7.58
b 5.74, −1.74
c 12.5, −0.481
d 6.24, 1.76
e 9.52, −0.525
f 6.14, −1.14

Since the quadratic formula is just a general version of completing the square, why complete the square?

- **To solve fairly simple quadratics** that won't factorise, it is often quicker than the formula.
- **To find a maximum or minimum value** (eg a max or min point on a graph) – see next page…

*Note: when a quadratic equation has no solutions, rearranging will quickly lead to the square root of a negative number, so it will be clear that no solutions can be found.

Completing the Square to find minimum or maximum values

Quadratic equations can be solved (ie find the points where the graph crosses the x axis) in many different ways, but to find a max or min value, you need completing the square.

Find the minimum value of $x^2 + 8x - 5$ and the value of x that gives this minimum.

Step 1: Complete the square (see previous page for details)

$(x+4)^2 - 21$ *This expression is identical to the original, just rearranged.*

Step 2: Find the x value that makes the squared bracket zero

$x + 4 = 0 \Longrightarrow \boldsymbol{x = -4}$ *Anything squared can never be lower than 0, so make it 0.*

Step 3: The number at the end is the minimum value

$\boldsymbol{-21}$ *When the squared bracket is 0, the quadratic is -21.*

It is not obvious that $x^2 + 8x - 5$ will never give a value below -21 for any value of x, but by rewriting the expression as $(x+4)^2 - 21$ it is clear what the minimum is, and the value of x that will give us this minimum value. If on a graph, the minimum point is $(-4, -21)$.

1. Find the minimum value, and the value of x that gives this minimum, for each of the quadratic expressions below:

a) $x^2 + 10x + 2$ b) $x^2 + 2x - 15$ c) $x^2 + 14x + 50$

d) $x^2 - 4x + 7$ e) $x^2 - 4x$ f) $x^2 + 5$

2. Find the minimum value, and the value of x that gives this minimum, for each of the quadratic expressions below:

a) $x^2 + 3x + 12$ b) $x^2 + 5x - 15$ c) $x^2 - x$

Finding a maximum point

Any quadratic with a positive x^2 will have a minimum, but if the x^2 coefficient is negative, it will have a maximum. When you complete the square, take out -1 as a common factor, then when you simplify and read off values, notice that the squared bracket has a – in front, so you find a maximum by taking away as little as possible.

1. a) $(x+5)^2 - 23$ has min -23 when $x = -5$
b) $(x+1)^2 - 16$ has min -16 when $x = -1$
c) $(x+7)^2 + 1$ has min 1 when $x = -7$
d) $(x-2)^2 + 3$ has min 3 when $x = -5$
e) $(x-2)^2 - 4$ has min -4 when $x = 2$
f) $x^2 + 5$ has min 5 when $x = 0$

2. a) $\left(x+\frac{3}{2}\right)^2 - \frac{39}{4}$ has min $-\frac{39}{4}$ when $x = -\frac{3}{2}$
b) $\left(x+\frac{5}{2}\right)^2 - \frac{85}{4}$ has min $-\frac{85}{4}$ when $x = -\frac{5}{2}$
c) $\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}$ has min $-\frac{1}{4}$ when $x = \frac{1}{2}$

Extension section: Completing the square for other coefficients of x^2

To convert *any* quadratic into completed square form, first take out the x^2 coefficient as a common factor of the entire expression, then multiply it back in at the end as needed.

Eg: $-2x^2 - 12x + 20 = -2[x^2 + 6x - 10] = -2[(x+3)^2 - 19] = \boldsymbol{-2(x-3)^2 + 38}$

This is a negative quadratic. Setting $x = 3$ makes the squared bracket 0, so the maximum value is 38. The graph of $y = -2(x-3)^2 + 38$ has maximum point (3,38).

Messy example with fractions:

$$3x^2 + 7x - 12 = 3\left[x^2 + \frac{7}{3}x - 4\right] = 3\left[\left(x+\frac{7}{6}\right)^2 - \frac{49}{36} - 4\right] = 3\left(x+\frac{7}{6}\right)^2 - \frac{49}{6} - 12 = 3\left(x+\frac{7}{6}\right)^2 - \frac{121}{6}$$

The quadratic has min value $-\frac{121}{6}$ when $x = -\frac{7}{6}$. The graph $y = 3\left(x+\frac{7}{6}\right)^2 - \frac{121}{6}$ has max $\left(-\frac{7}{6}, -\frac{121}{6}\right)$.